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Geometrie der Weltkartenentwürfe mit computergraphischer Darstellung

Inhaltsübersicht

- Einführung in die Differentialgeometrie
- Echte Entwürfe
- Unechte Entwürfe
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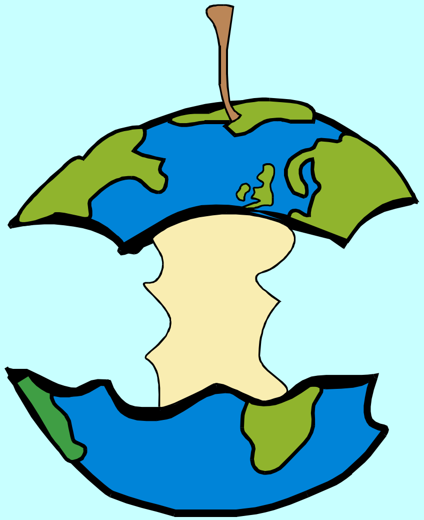
- Mischkarten und umbezifferte Entwürfe

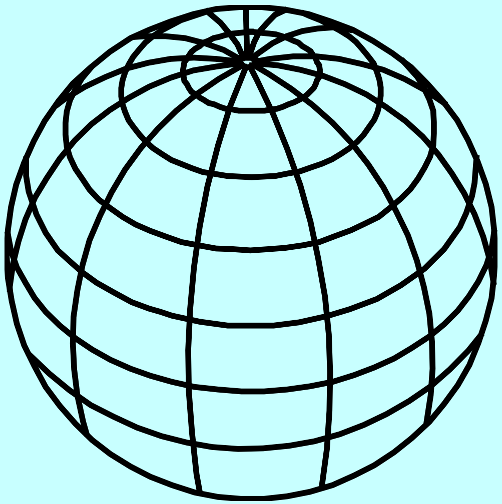


Kugelkoordinaten

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r & \cos(v) & \cos(u) \\ r & \cos(v) & \sin(u) \\ r & \sin(v) & \end{pmatrix}$$

$$u \in (-\pi, \pi]$$

$$v \in [-\pi/2, \pi/2]$$

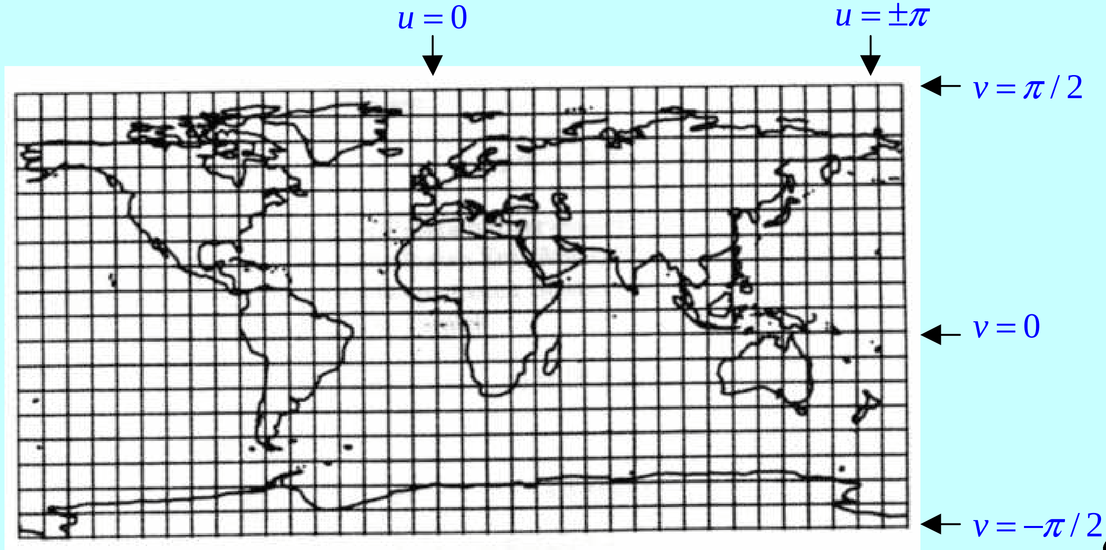


Die Linien mit $v = \text{konstant}$

heißen **Breitengrade**

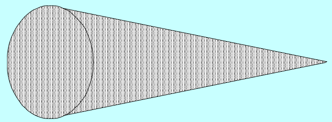
Die Linien mit $u = \text{konstant}$

heißen **Längengrade**



Kegel / Zylinderabwicklung

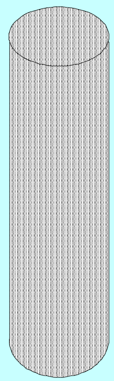
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} v \sin\theta \cos u \\ v \sin\theta \sin u \\ v \cos\theta + \sigma u \end{pmatrix} \qquad u \in (-\pi, \pi] \qquad v \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$



$$y_1 = v \sin(u \sin\theta) \qquad y_2 = -v \cos(u \sin\theta)$$

$\theta =$ Öffnungswinkel des Kegels

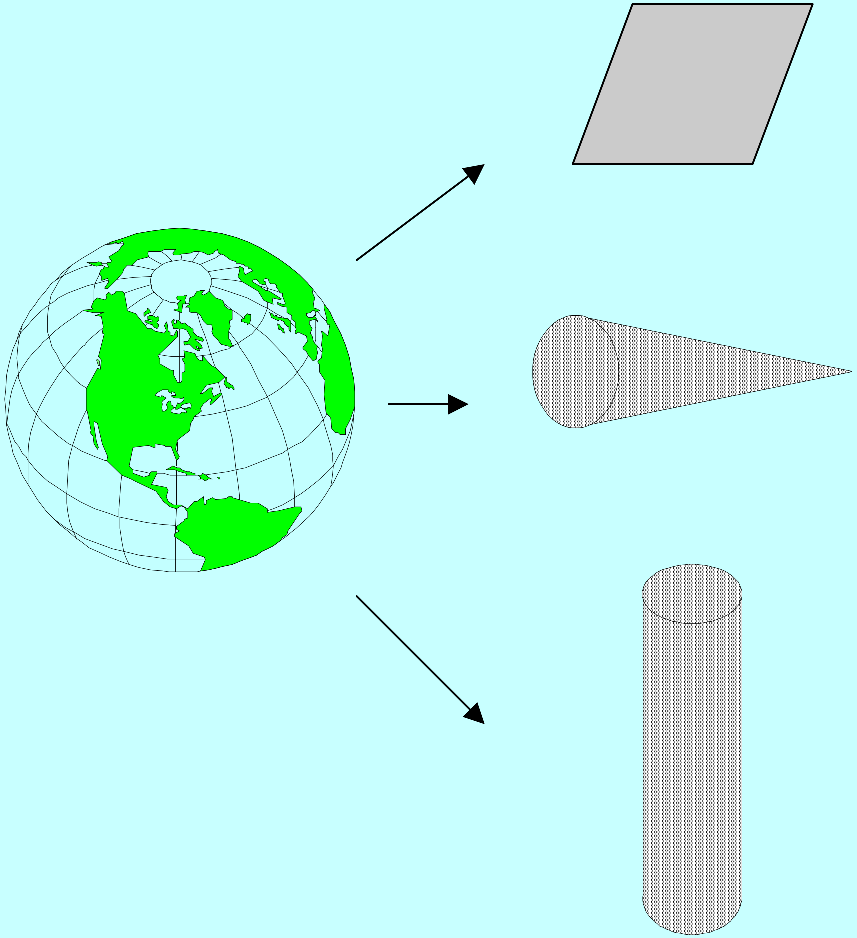
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos u \\ r \sin u \\ v \end{pmatrix} \qquad u \in (-\pi, \pi] \qquad v \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

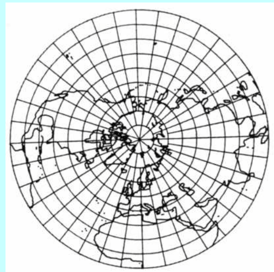


$$y_1 = r\, u \qquad y_2 = v$$

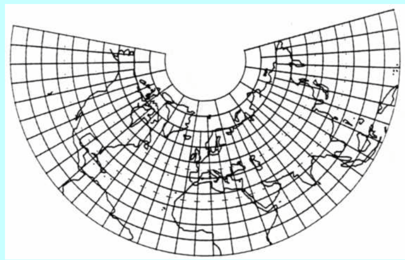
Echte Entwürfe

Ein Entwurf heißt echt $\Leftrightarrow$ die Längen und Breitengrade werden auf Kreis bzw. Geradenstücke abgebildet

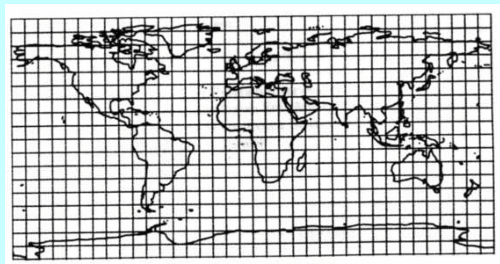




$$x = \; r(v)\sin(u)$$
$$y = -\,r(v)\cos(u)$$



$$x = \; r(v)\sin(ku)$$
$$y = -\,r(v)\cos(ku)$$
$$(k = \sin\theta)$$



$$x = \; k\,u$$
$$y = \; r(v)$$

Fundamentalgrößen

Sei $\Phi \subseteq \Re^3$ eine Fläche mit regulärer Parameterdarstellung $\vec{x} = \vec{x}(u,v)$, dann heißen

$$E(\vec{x}) := \vec{x}'^{u} \cdot \vec{x}'^{u} \qquad F(\vec{x}) := \vec{x}'^{u} \cdot \vec{x}'^{v} \qquad G(\vec{x}) := \vec{x}'^{v} \cdot \vec{x}'^{v}$$

die *ersten Fundamentalgrößen* von Φ.

Bei der Sphäre lauten die ersten Fundametalgrößen

$$E(\vec{x}) = \cos^2 v \qquad F(\vec{x}) = 0 \qquad G(\vec{x}) = 1$$

Bei Kegelentwürfen lauten die Fundamentalgrößen

$$E(\vec{y}) = (r(v) \cdot k)^2 \qquad F(\vec{y}) = 0 \qquad \mathrm{G}(\vec{y}) = (r'(v))^2$$

Die Flächenverzerrung der Parametrisierung ist :

$$|\, \vec{x}'^{u} \times \vec{x}'^{v} \,| = \sqrt{(\vec{x}'^{u})^2 (\vec{x}'^{v})^2 - (\vec{x}'^{u} \vec{x}'^{v})^2 \,|} = \sqrt{E(\vec{x}) G(\vec{x}) - F^2(\vec{x})}$$

Abbildungsinvarianten

Eine Abbildung $\vec{y} = f(\vec{x})$ heißt

abstandstreu $\Leftrightarrow$ die Meridiane werden längentreu abgebildet

$\Leftrightarrow \lambda_v := \sqrt{\dfrac{G(\vec{y})}{G(\vec{x})}} = 1.\ \lambda_v$ heißt Verzerrung in v Richtung

abweitungstreu $\Leftrightarrow$ die Breitengrade werden längentreu abgebildet

$\Leftrightarrow \lambda_u := \sqrt{\dfrac{E(\vec{y})}{E(\vec{x})}} = 1.\ \lambda_u$ heißt Verzerrung in u Richtung

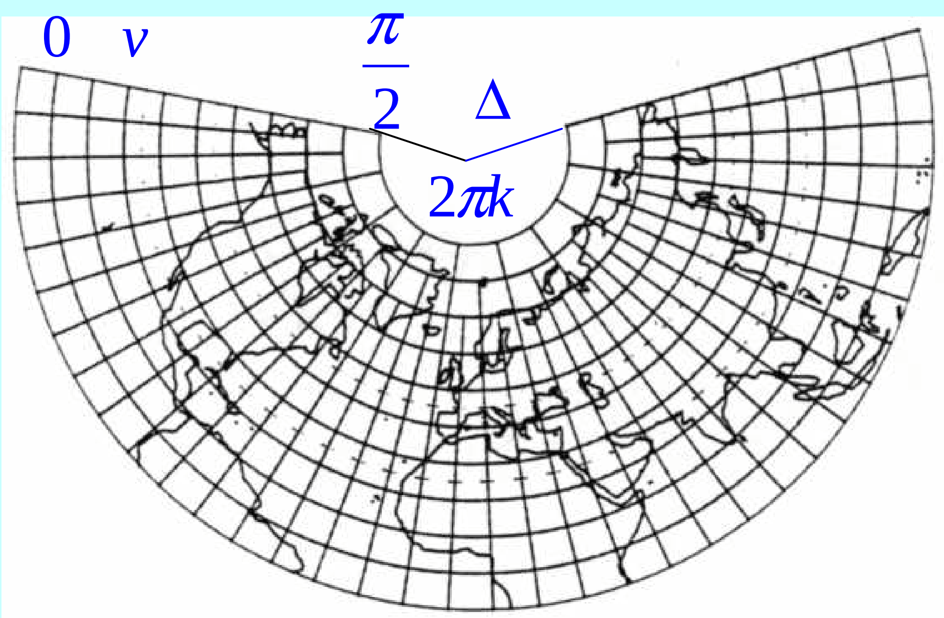
flächentreu $\Leftrightarrow E(\vec{x})G(\vec{x}) - F^2(\vec{x}) = E(\vec{y})G(\vec{y}) - F^2(\vec{y})$

winkeltreu $\Leftrightarrow E(\vec{x}) = g(u,v)E(\vec{y}), F(\vec{x}) = g(u,v)F(\vec{y}), G(\vec{x}) = g(u,v)G(\vec{y}) \Leftrightarrow \dfrac{E(\vec{y})}{E(\vec{x})} = \dfrac{G(\vec{y})}{G(\vec{x})}$

Abstandstreue Kegelentwürfe

2 Breitengrade $v_1 = \text{konst.}$ und $v_2 = \text{konst.}$ $(v_1 \neq v_2)$ sollen längentreu abgebildet werden:

$$2 \cdot \pi \cdot \cos v_i = 2 \cdot \pi \cdot k \cdot (\Delta + \pi / 2 - v_i) \quad i = 1{,}2$$



$$k = \frac{\cos v_1 - \cos v_2}{v_2 - v_1}$$

$$\Delta = \frac{\cos v_2}{k} - \frac{\pi}{2} + v_2$$

$$r(v) = \Delta + \frac{\pi}{2} - v_2$$

$$y_1 = \left(\frac{\cos v_2 (v_2 - v_1)}{\cos v_1 - \cos v_2} - (v - v_2) \right) \sin\left(\frac{\cos v_1 - \cos v_2}{v_2 - v_1} u\right)$$

$$y_2 = \left(\frac{\cos v_2 (v_2 - v_1)}{\cos v_1 - \cos v_2} - (v - v_2) \right) \cos\left(\frac{\cos v_1 - \cos v_2}{v_2 - v_1} u\right)$$

Sonderfälle abstandstreuer Kegelentwürfe

Sei $v_1 (\neq 0, \pi/2)$

$$v_2 \to v_1 \;\square\; k = \frac{\cos v_1 - \cos v_2}{v_2 - v_1} \to \sin v_1 \,,$$

$$y_1 = (\cot v_1 - v + v_1) \sin(\sin v_1 \, u)$$

$$\Delta = \frac{\cos v_2}{k} - \frac{\pi}{2} + v_2 \to \cot v_1 - \frac{\pi}{2} + v_1$$

$$y_2 = -(\cot v_1 - v + v_1) \cos(\sin v_1 \, u)$$

$$v_1 \to \frac{\pi}{2} \;\square\; k = \sin v_1 \to 1 \,,$$

$$y_1 = (\frac{\pi}{2} - v) \sin u$$

$$\Delta = \cot v_1 - \frac{\pi}{2} + v_1 \to 0$$

$$y_2 = -(\frac{\pi}{2} - v) \cos u$$

$$v_2 \to -v_1 \neq \pm \tfrac{\pi}{2} \;\square\; k = \frac{\cos v_1 - \cos v_2}{v_2 - v_1} \to 0 \,,$$

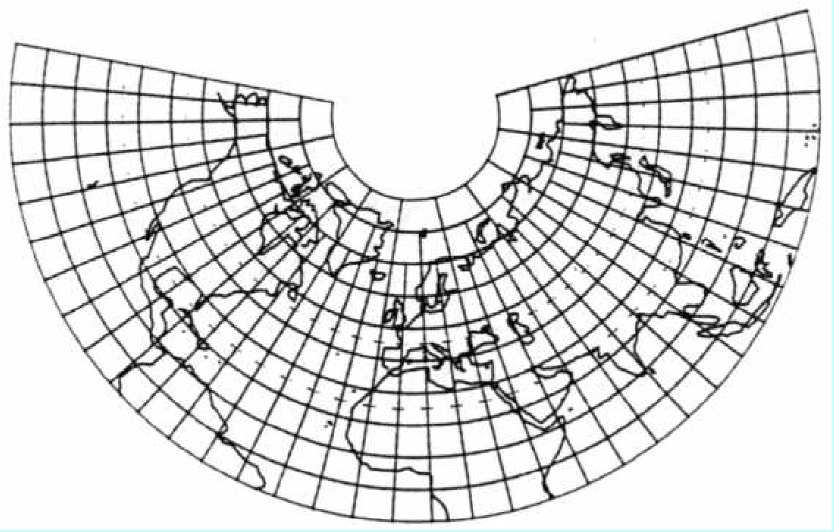
$$y_1 = (\frac{\cos v_1}{k} - v) \sin(k\, u) \to u \cos v_1$$

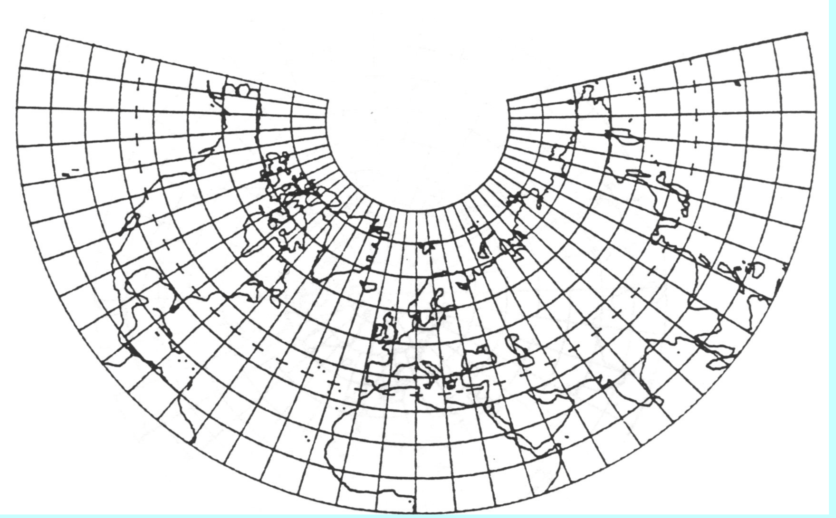
$$\Delta = \frac{\cos v_1}{k} - \frac{\pi}{2} + v_1 \to \infty$$

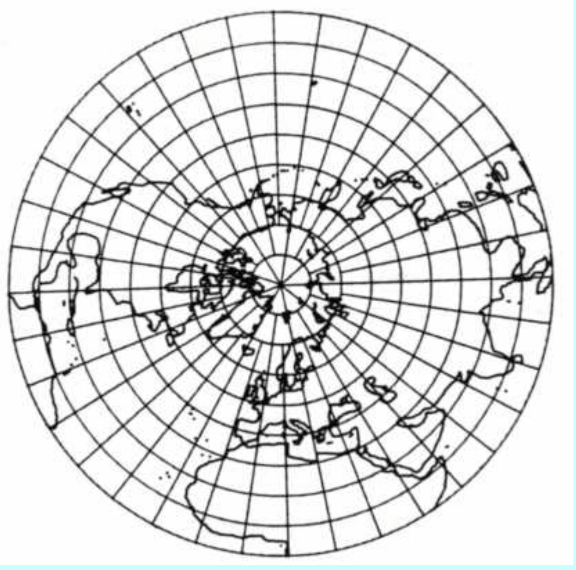
$$y_2 = -(\frac{\cos v_1}{k} - v) \cos(k\, u) \to v - \infty$$

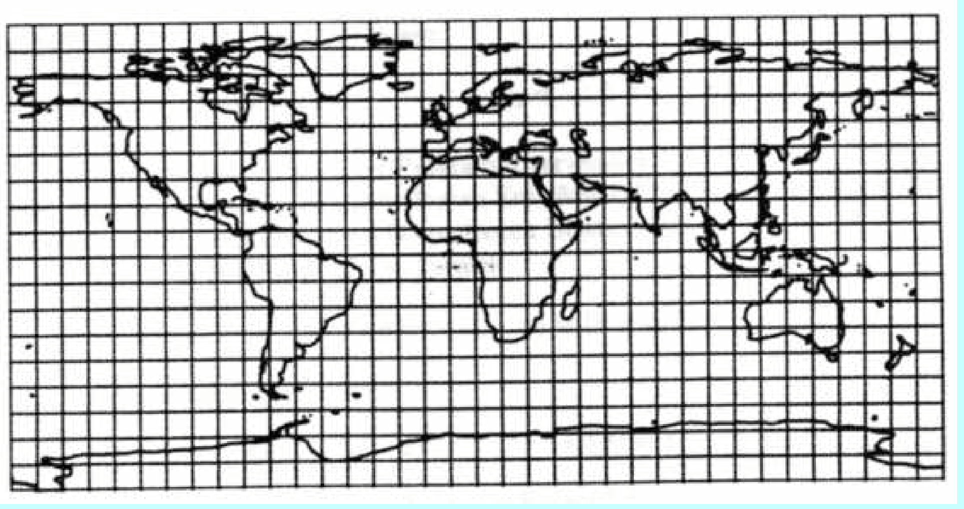
$$\eta_2 = y_2 + \frac{\cos v_1}{k} \to v$$

Abstandstreue Entwürfe









Abweitungstreue Kegelentwürfe

Ansatz für Abweitungstreue : $E^2(\vec{y}) = E^2(\vec{x}) \;\square\; (r(v)\sin\theta)^2 = \cos^2 v$

$$y_1 = \frac{\cos v}{\sin\theta}\sin(u\sin\theta) \qquad y_2 = \frac{-\cos v}{\sin\theta}\cos(u\sin\theta)$$

(Parallelprojektion auf einen Kegel)

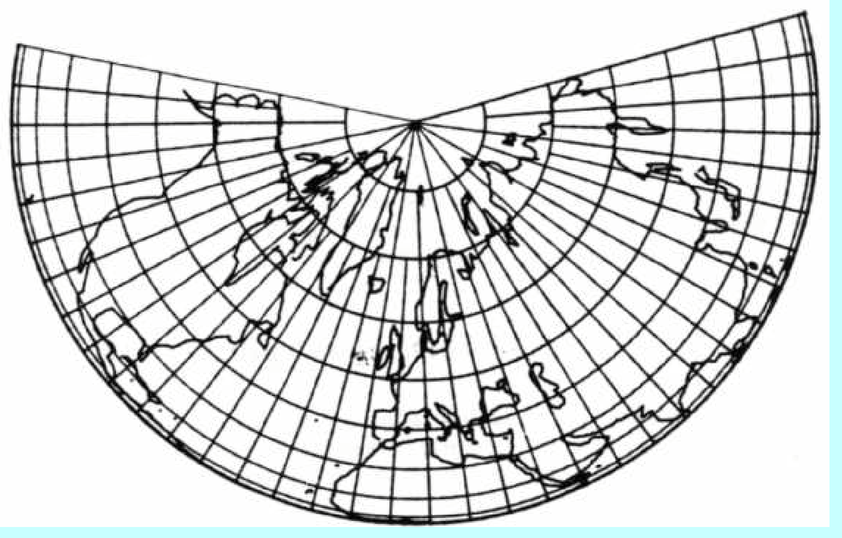
Sonderfall : $\theta \to \frac{\pi}{2}$ $\qquad \sin\theta \to 1$

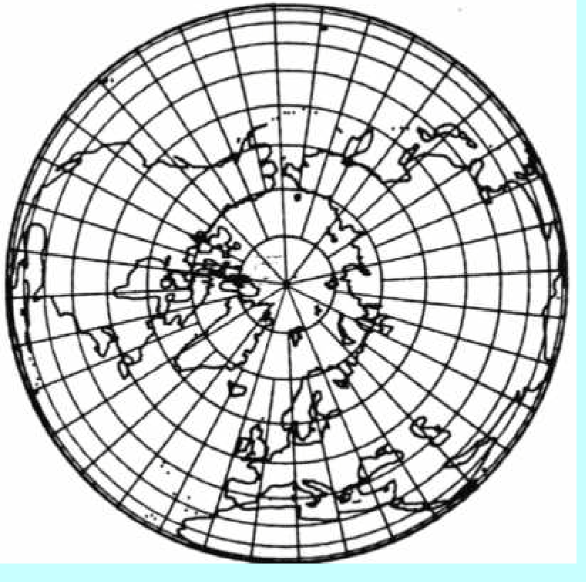
$$y_1 = \cos v \sin u \qquad y_2 = -\cos v \cos u$$

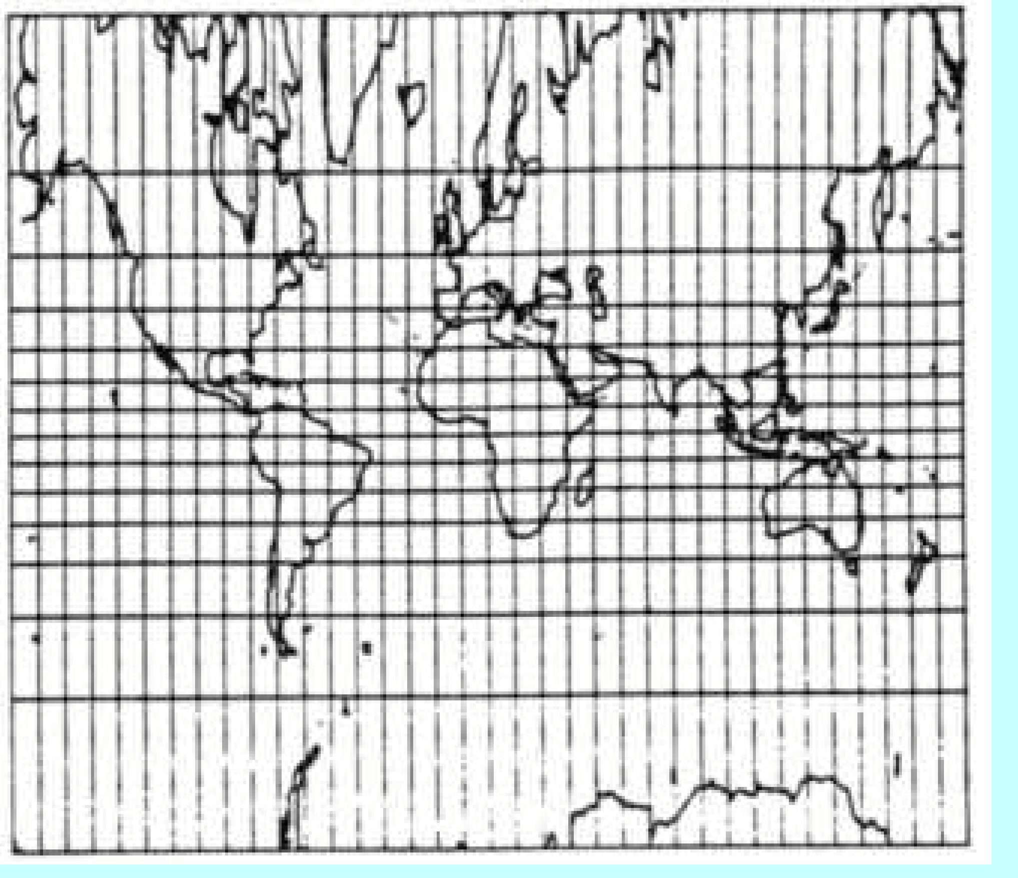
(Parallelprojektion in die Ebene)

Abweitungstreue Zylinderentwürfe sind unmöglich

Projektionen







Flächentreue Kegelentwürfe

Ein Entwurf ist *flächentreu* $\Leftrightarrow E(\vec{x})G(\vec{x}) - F^2(\vec{x}) = E(\vec{y})G(\vec{y}) - F^2(\vec{y})$

$(r(v)\cdot k)^2 (r'(v))^2 = \cos^2 v \quad \Leftrightarrow \frac{1}{2} r^2(v) = \pm \frac{\sin v}{k} + c \quad c, r(v), k$ noch zu best.

v_1 und v_2 sollen längentreu sein: $2\pi k r(v_i) = 2\pi \cos v_i \quad$ für $i = 1,2$

$$k = \frac{\sin v_1 + \sin v_2}{2}, \quad c = \pm 2 \frac{\sin v_1}{k} + \frac{\cos^2 v_1}{v^2}, \quad r(v) = 2 \frac{\sqrt{\cos^2 v_1 + (\sin v_1 - \sin v)(\sin v_1 + \sin v_2)}}{(\sin v_1 + \sin v_2)}$$

$$y_1 = 2 \frac{\sqrt{\cos^2 v_1 + (\sin v_1 - \sin v)(\sin v_1 + \sin v_2)}}{(\sin v_1 + \sin v_2)} \sin\left(\frac{\sin v_1 + \sin v_2}{2} u\right)$$

$$y_2 = -2 \frac{\sqrt{\cos^2 v_1 + (\sin v_1 - \sin v)(\sin v_1 + \sin v_2)}}{(\sin v_1 + \sin v_2)} \cos\left(\frac{\sin v_1 + \sin v_2}{2} u\right)$$

Sonderfälle flächentreuer Kegelentwürfe

$$v_2 \to v_1 \;\square\; k \to \sin v_1 \,,$$

$$r(v) \to \frac{\sqrt{1+\sin^2 v_1 - 2\sin v \sin v_1}}{\sin v_1}$$

$$y_1 = \frac{\sqrt{1+\sin^2 v_1 - 2\sin v \sin v_1}}{\sin v_1} \sin\left(u \sin v_1\right)$$

$$y_2 = \frac{\sqrt{1+\sin^2 v_1 - 2\sin v \sin v_1}}{\sin v_1} \cos\left(u \sin v_1\right)$$

$$v_1 \to \frac{\pi}{2} \;\square\; k \to 1, r(v) \to \sqrt{2(1-\sin v)}$$

(Entwurf nach Lambert)

$$y_1 = \sqrt{2(1-\sin v)}\,\sin u$$

$$y_2 = -\sqrt{2(1-\sin v)}\,\cos u$$

$$v_2 \to -v_1 \neq \pm\pi/2 :$$

$$k = \frac{\sin v_1 + \sin v_2}{2} \to 0,$$

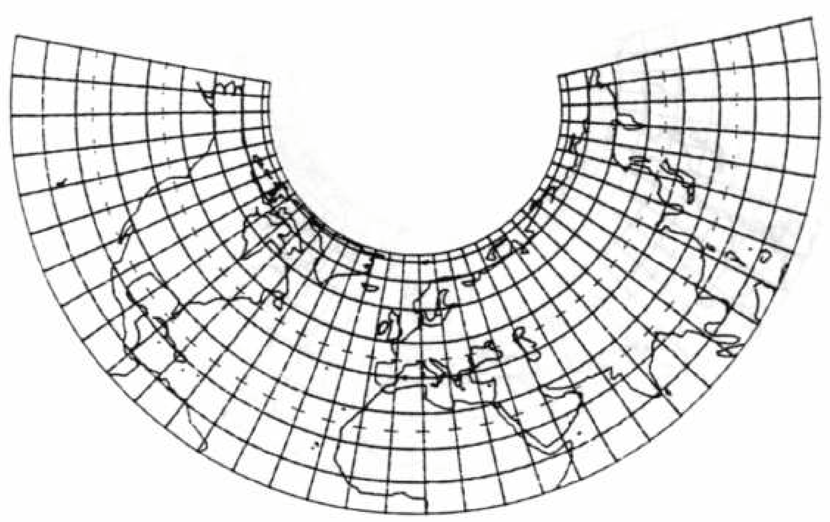
$$r(v) = 2\frac{\sqrt{\cos^2 v_1 + (\sin v_1 - \sin v)(\sin v_1 + \sin v_2)}}{(\sin v_1 + \sin v_2)}$$

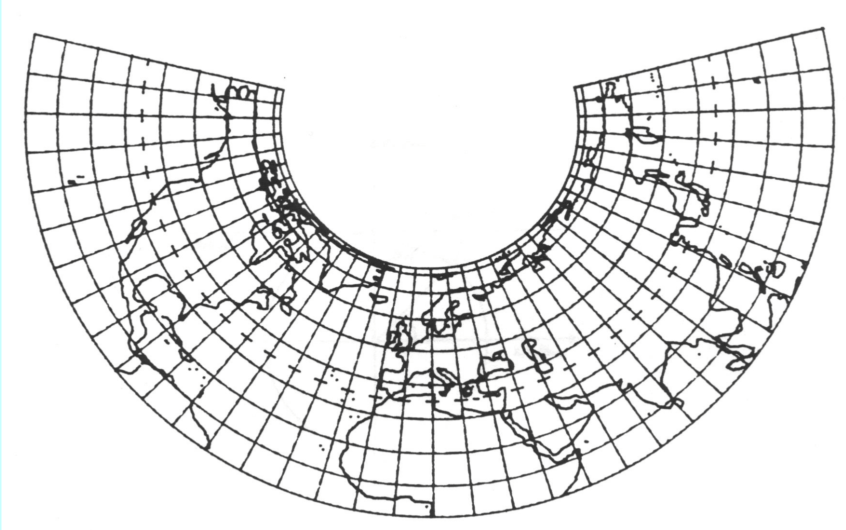
$$\to \infty$$

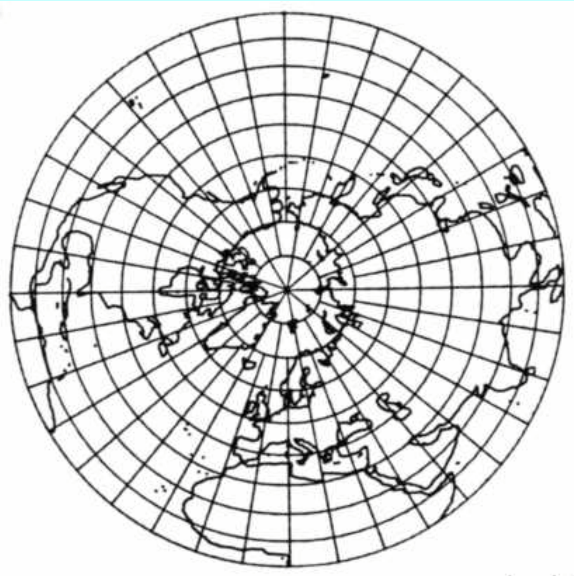
$$y_1 = u \cos v_1$$

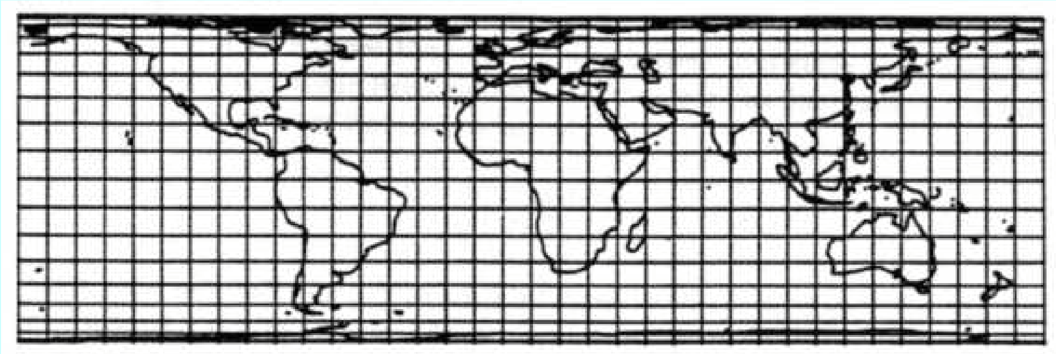
$$\eta_2 = \frac{\sin v}{\cos v_1}$$

Flächentreue Entwürfe









Winkeltreue Kegelentwürfe

$$\text{Ansatz für Winkeltreue}: \frac{E(\vec{y})}{E(\vec{x})} = \frac{G(\vec{y})}{G(\vec{x})} \;\square\; \frac{(r(v)k)^2}{\cos^2 v} = \frac{(r'(v))^2}{1} \;\square\; \frac{r'(v)}{r(v)} = \frac{\pm k}{\cos v}$$

$$\square\; \ln|r(v)| = \pm k \ln(\tan(\frac{\pi}{4} + \frac{v}{2})) + c' \;\square\; r(v)) = c \tan^{\pm k}(\frac{\pi + 2v}{4})$$

(das ± vertauscht Nord und Südpol)

Die Breitengrade $v = v_i$ werden längentreu abgebildet: $k\, r(v_i) = \cos v_i \quad i = 1{,}2$

$$\mathrm{c} = \frac{\cos v_1 \tan^k(\frac{\pi + 2v_1}{4})}{k} \qquad k = \frac{\ln \frac{\cos v_2}{\cos v_1}}{\ln \frac{\tan \frac{\pi + 2v_1}{4}}{\tan \frac{\pi + 2v_2}{4}}}$$

$$y_1 = \frac{\cos v_1 \tan^k(\frac{\pi + 2v}{4})}{k \tan^k(\frac{\pi + 2v}{4})} \sin(k\,u) \qquad y_1 = \frac{-\cos v_1 \tan^k(\frac{\pi + 2v}{4})}{k \tan^k(\frac{\pi + 2v}{4})} \cos(k\,u)$$

Sonderfälle winkeltreuer Kegelentwürfe

$$v_2 \to v_1 (\neq 0, \frac{\pi}{2}) \square \quad k \to \sin v_1 \quad \text{und} \qquad c \to \cot v_1 \tan^{\sin v_1}(\frac{\pi + 2v}{4})$$

$$y_1 = \cot v_1 \tan^{\sin v_1}(\frac{\pi + 2v_1}{4}) \cot^{\sin v_1}(\frac{\pi + 2v}{4}) \quad \sin(u \sin v_1)$$

$$y_2 = \cot v_1 \tan^{\sin v_1}(\frac{\pi + 2v_1}{4}) \cot^{\sin v_1}(\frac{\pi + 2v}{4}) \quad \cos(u \sin v_1)$$

$$v_1 \to \frac{\pi}{2}:$$

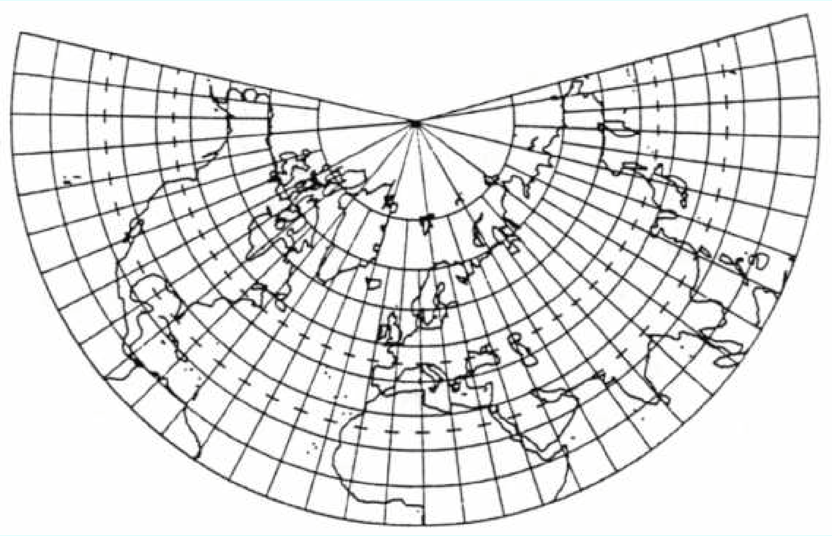
$$y_1 = 2\cot(\frac{\pi + 2v}{4}) \sin u \qquad y_2 = -2\cot(\frac{\pi + 2v}{4}) \cos u$$

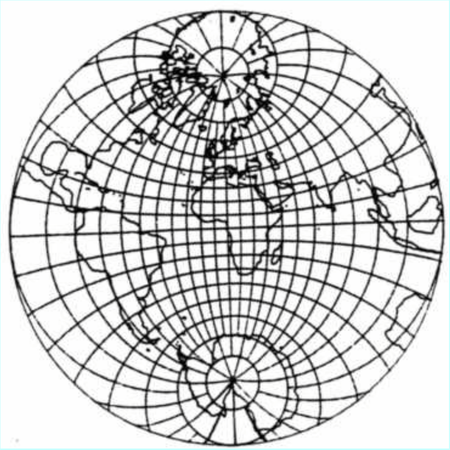
$$v_1 \to -v_2:$$

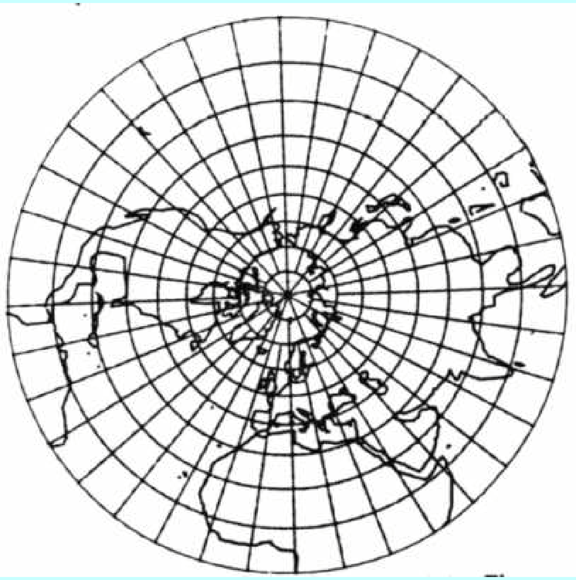
(Merkatorkarte)

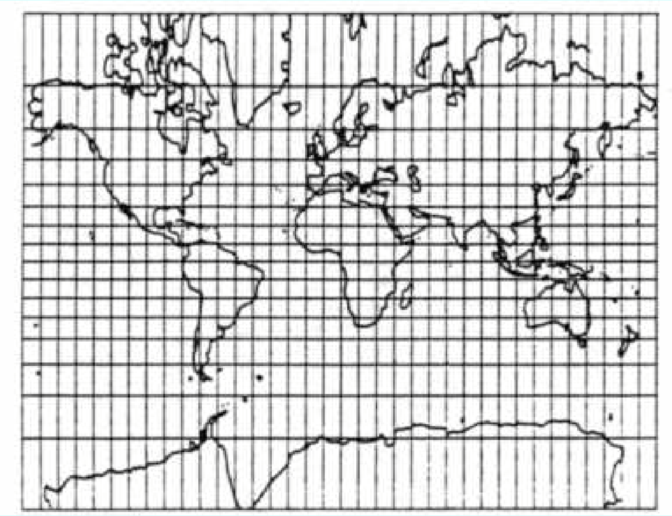
$$y_1 = u \cos v_1 \qquad \eta_2 = \cos v_1 \ln(\tan(\frac{\pi + 2v}{4}))$$

Winkeltreue Entwürfe

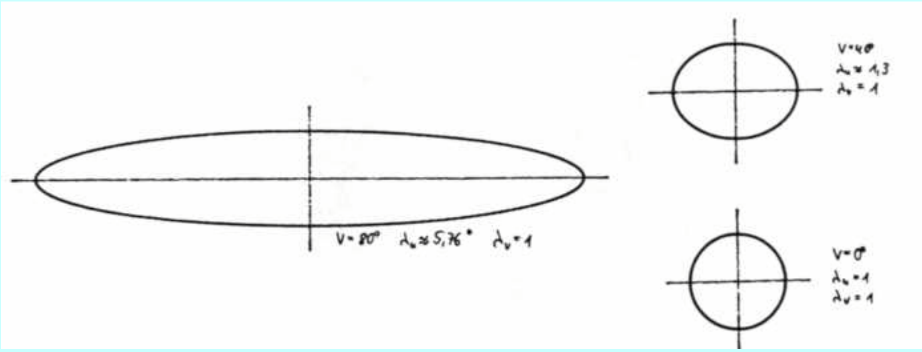


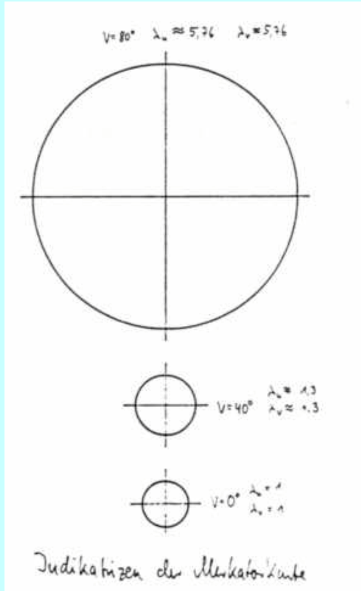


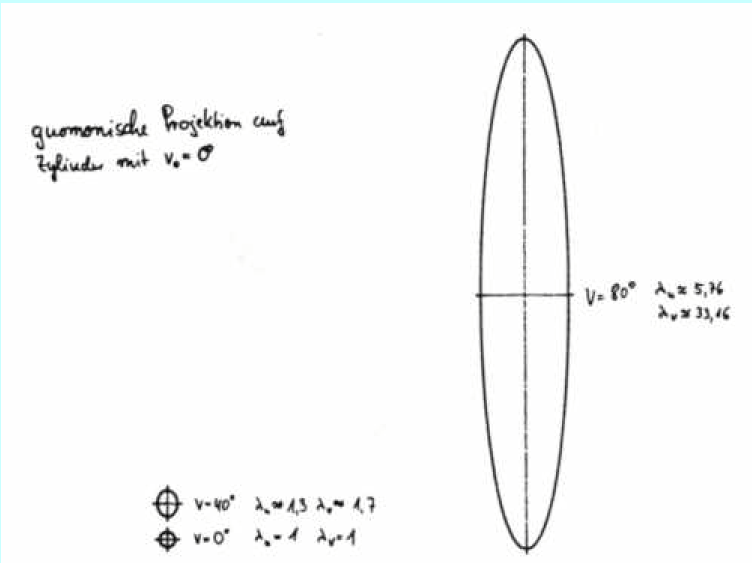


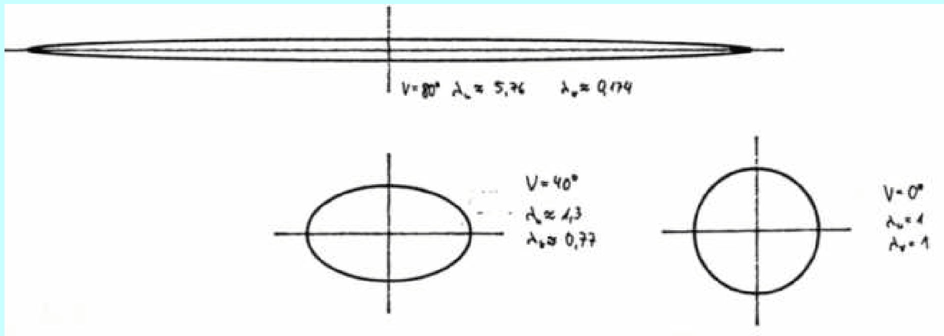


Tissotsche Indikatrix

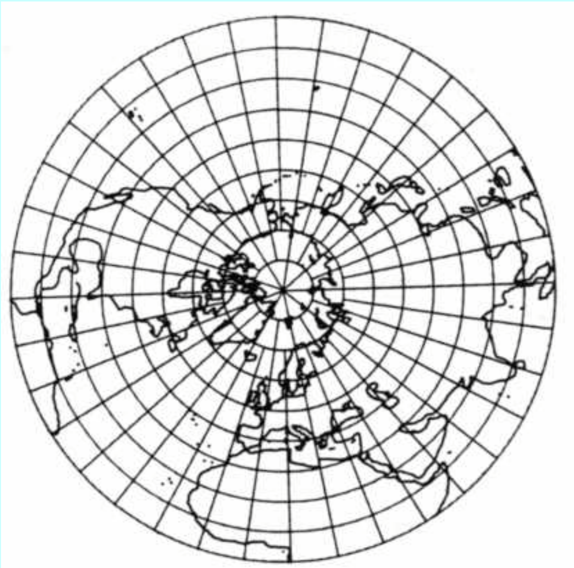


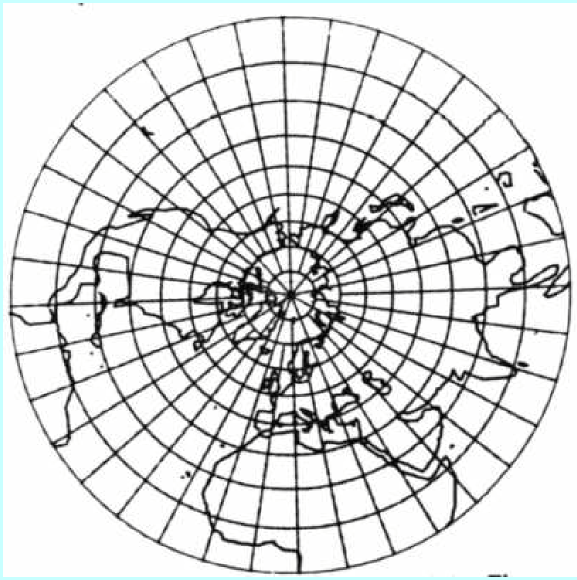


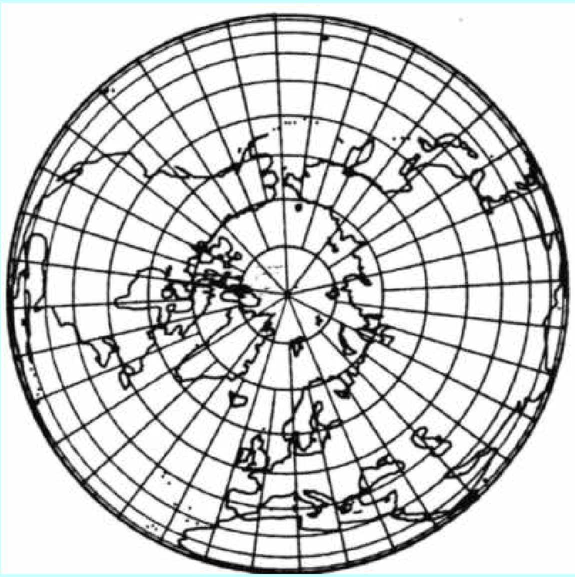


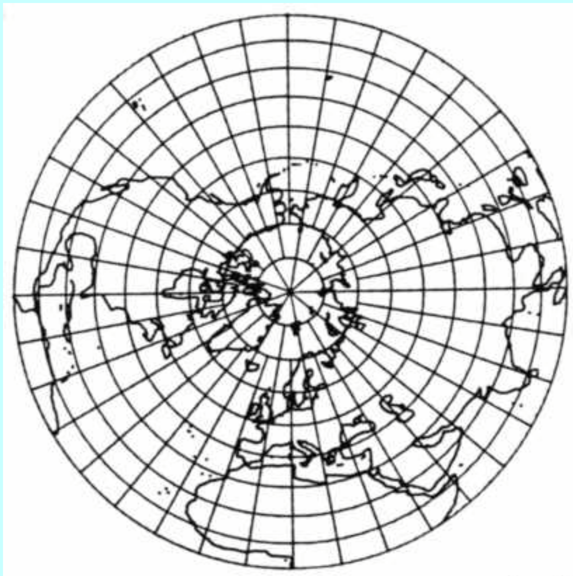


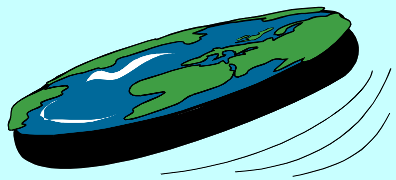
Azimutale Entwürfe



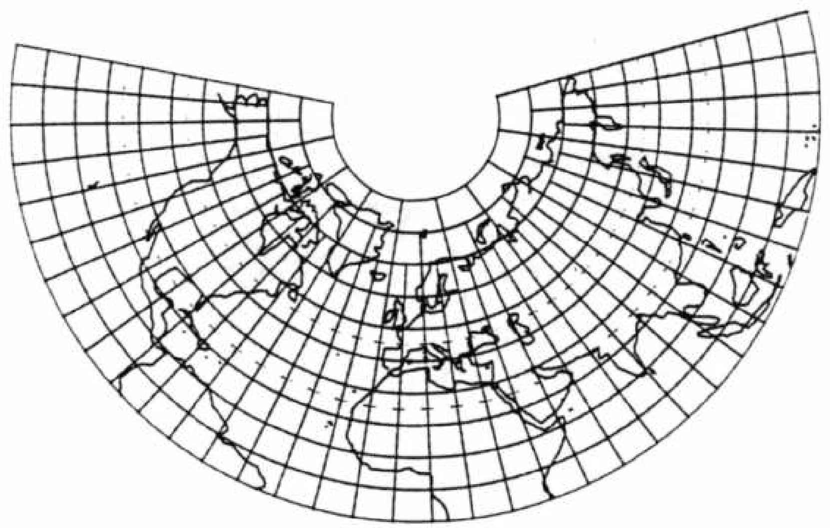
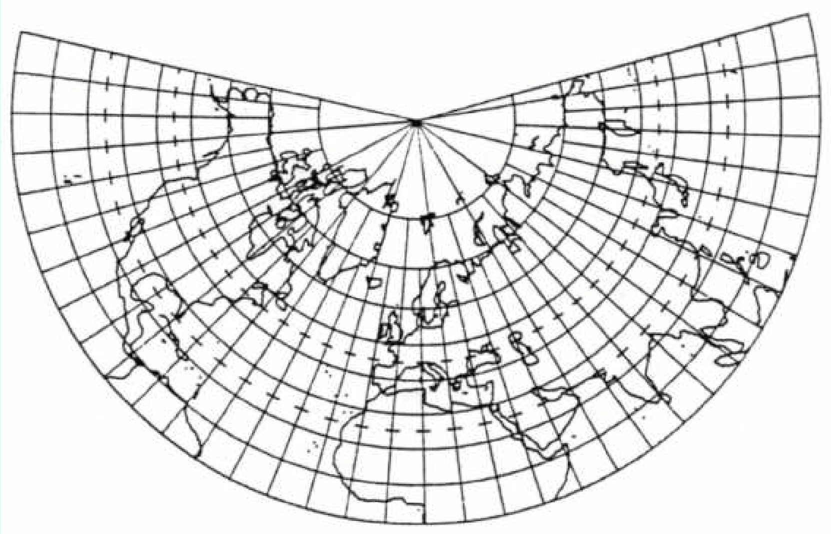
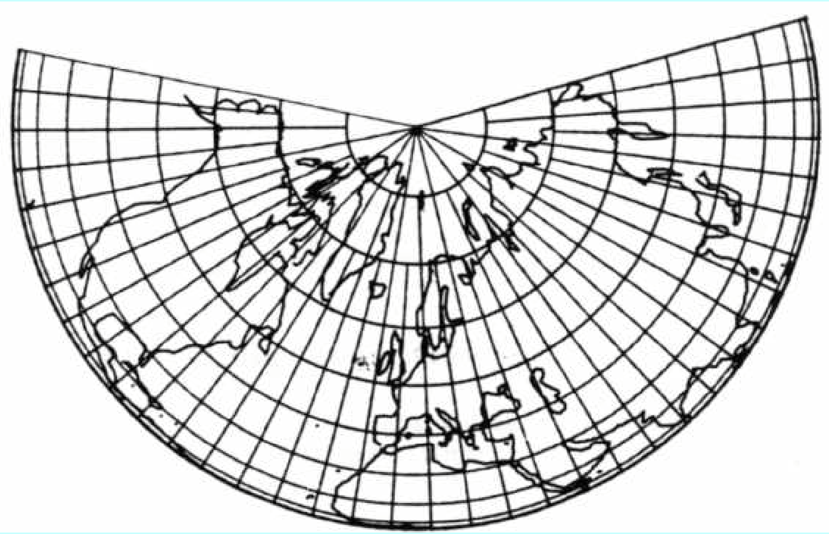
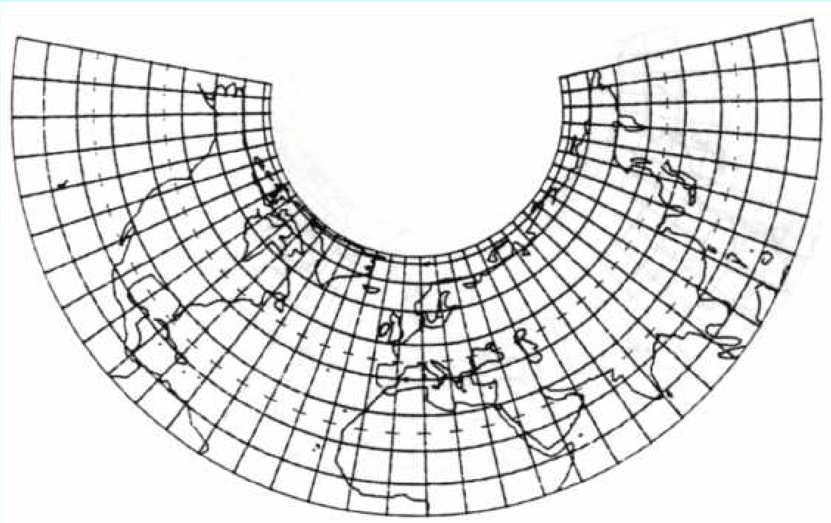
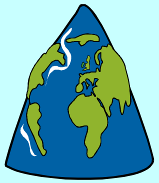




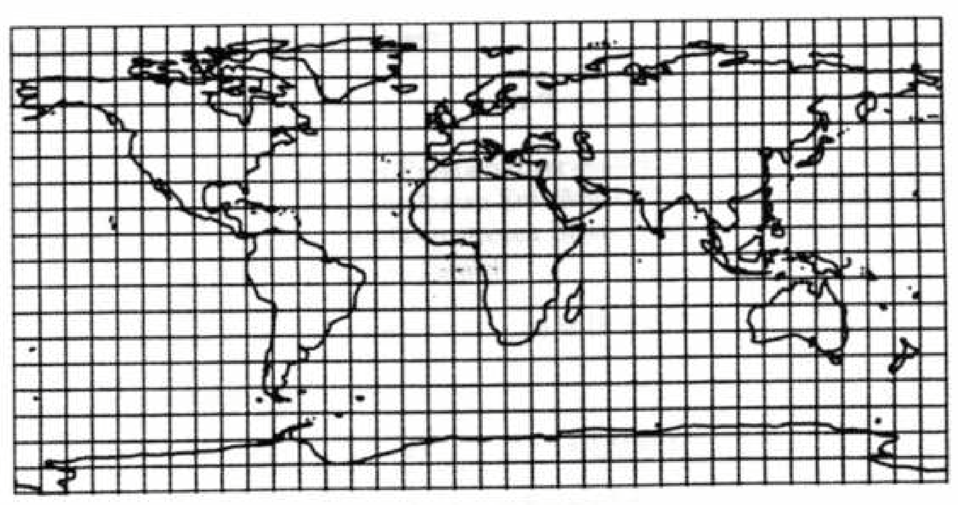


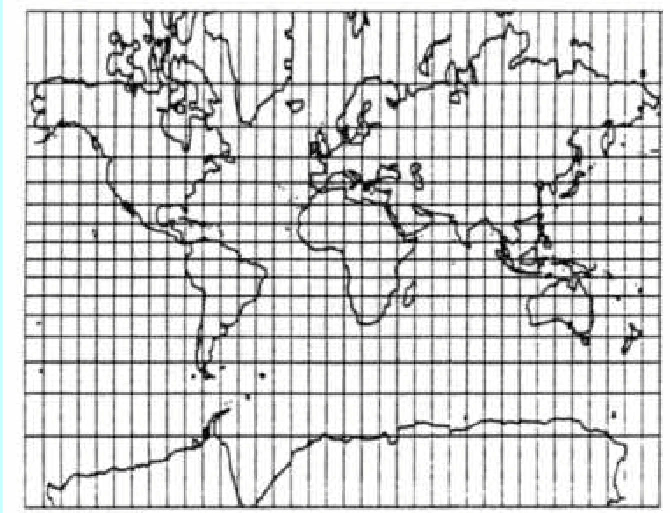


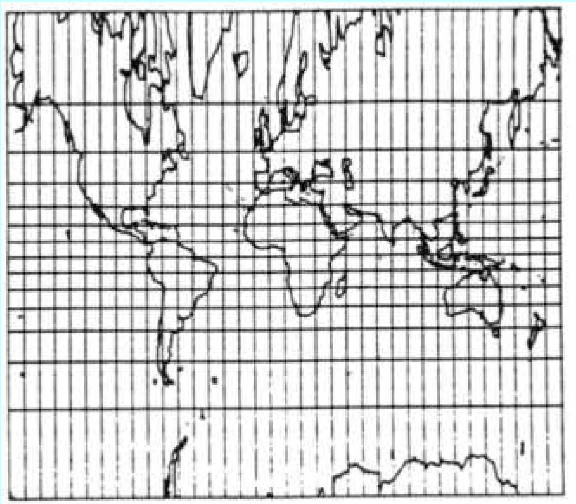
Kegelentwürfe

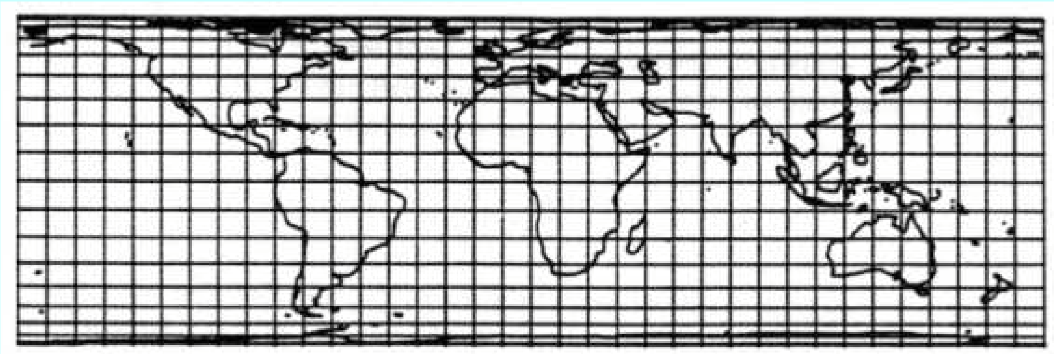






Zylinderentwürfe

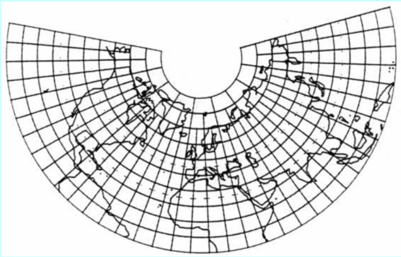




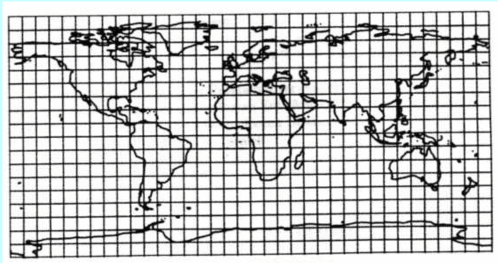




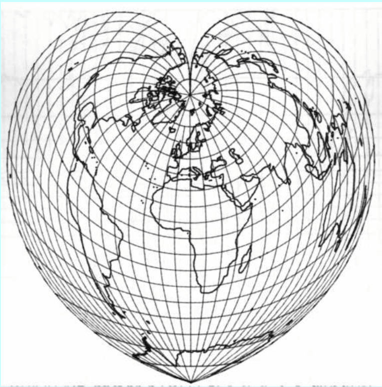
Unechte Entwürfe



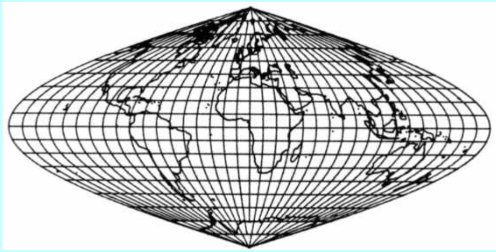
$$x = \; r(v)\sin(ku)$$
$$y = -\,r(v)\cos(ku)$$
$$(k = \sin\theta)$$



$$x = \; k\,u$$
$$y = \; r(v)$$

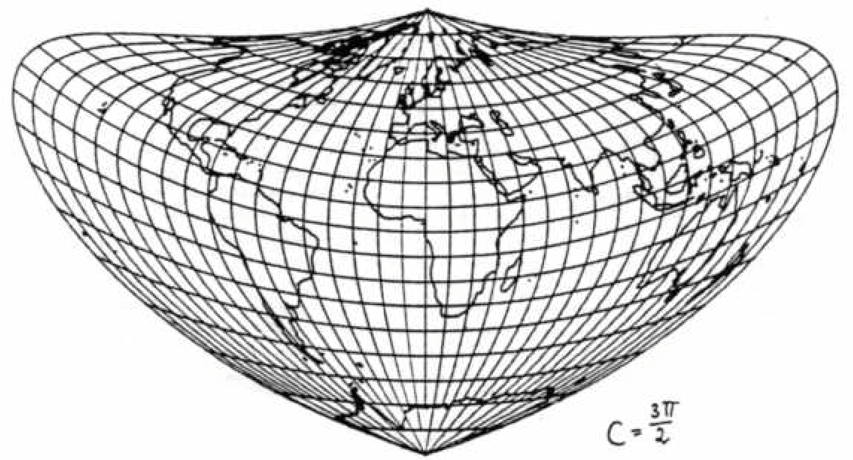


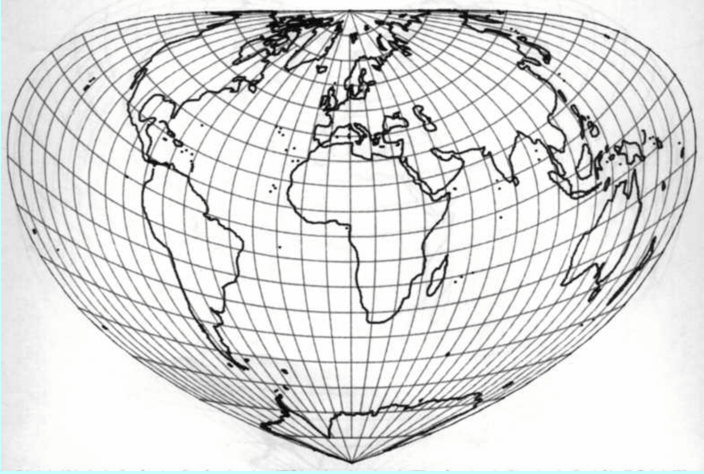
$$x = \; r(v)\sin(k(v)u)$$
$$y = -\,r(v)\cos(k(v)u)$$

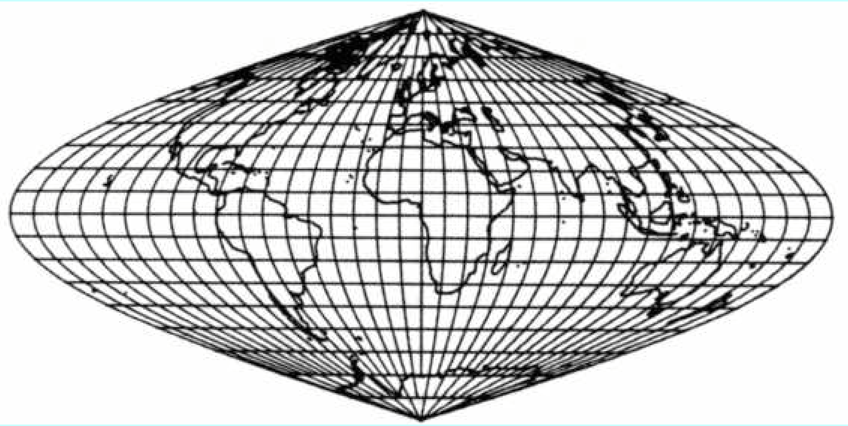


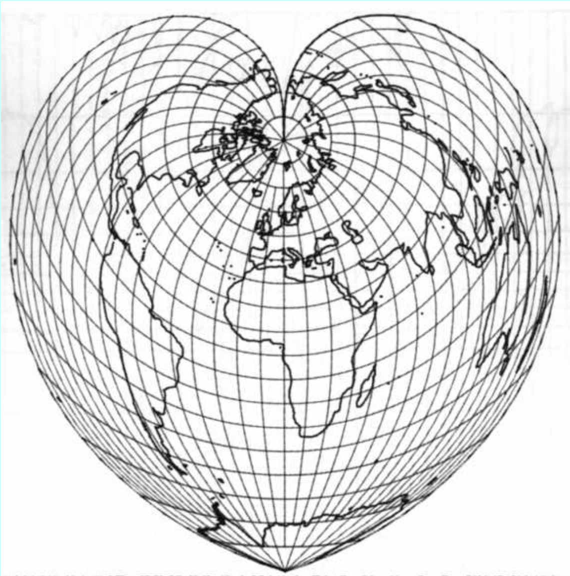
$$x = \; k(v)\,u$$
$$y = \; r(v)$$

Bonne Entwürfe

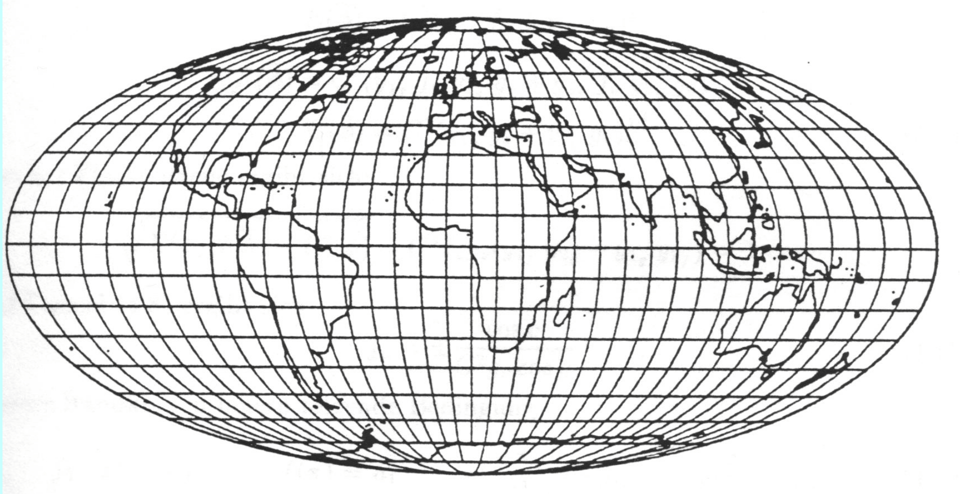




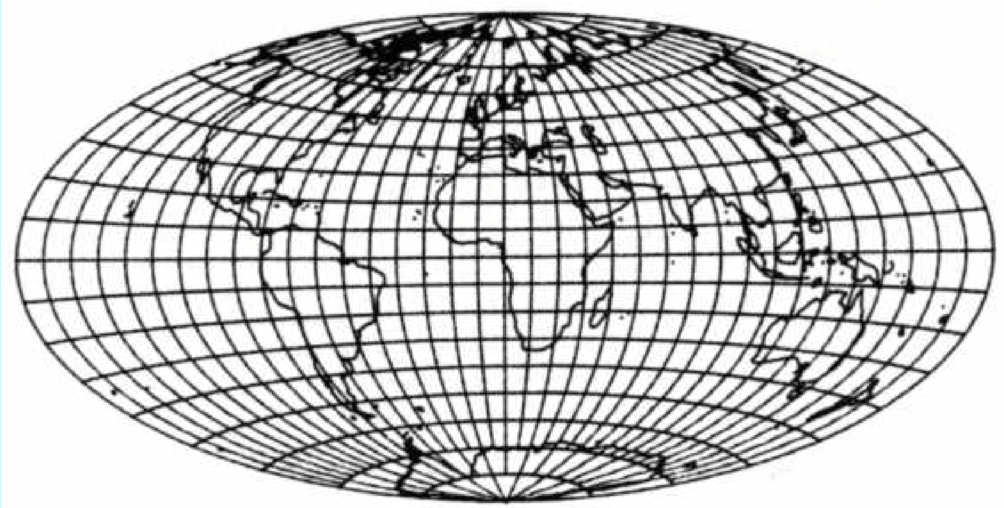




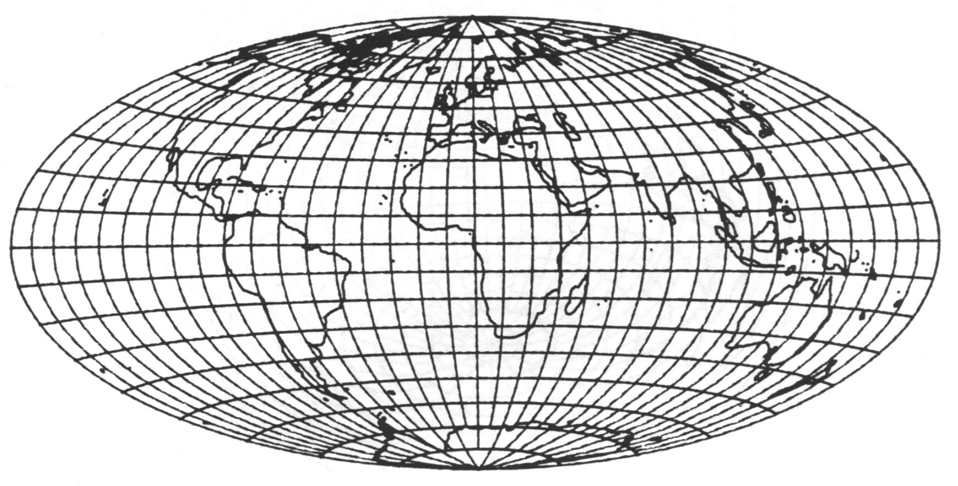
Unechte Entwürfe



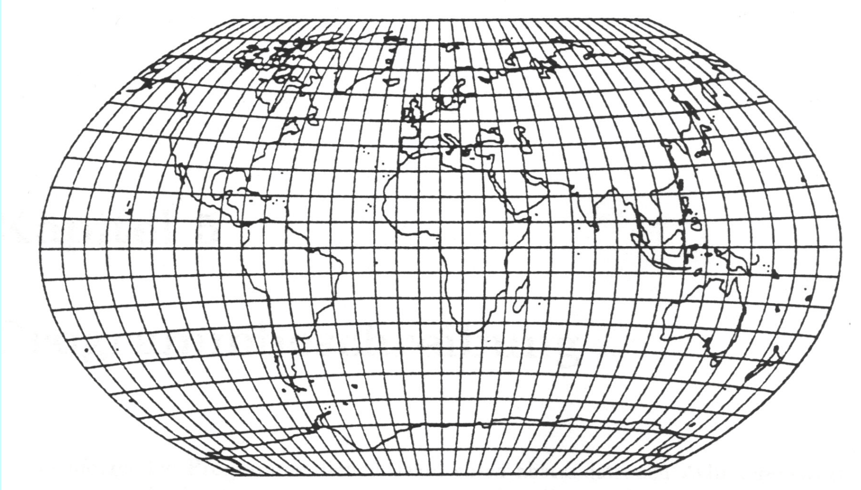
Mollweide



Aitoff

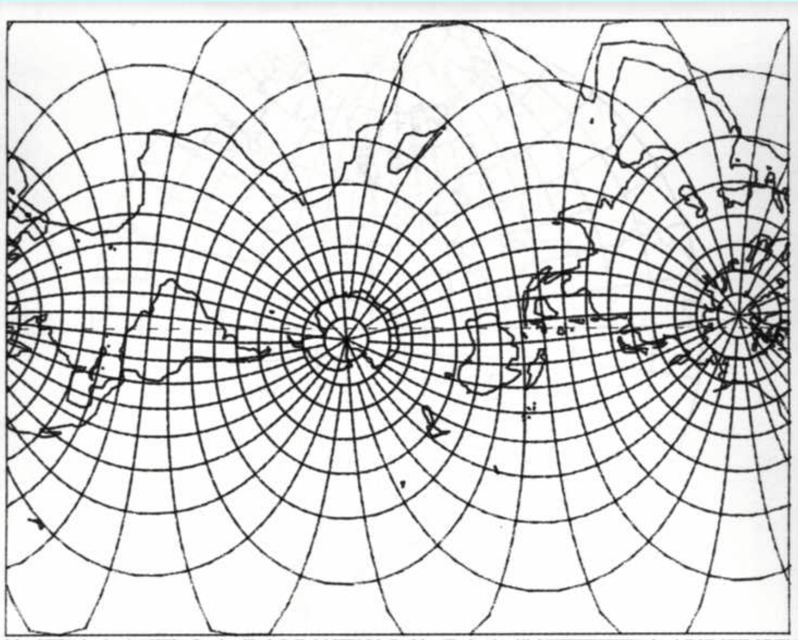


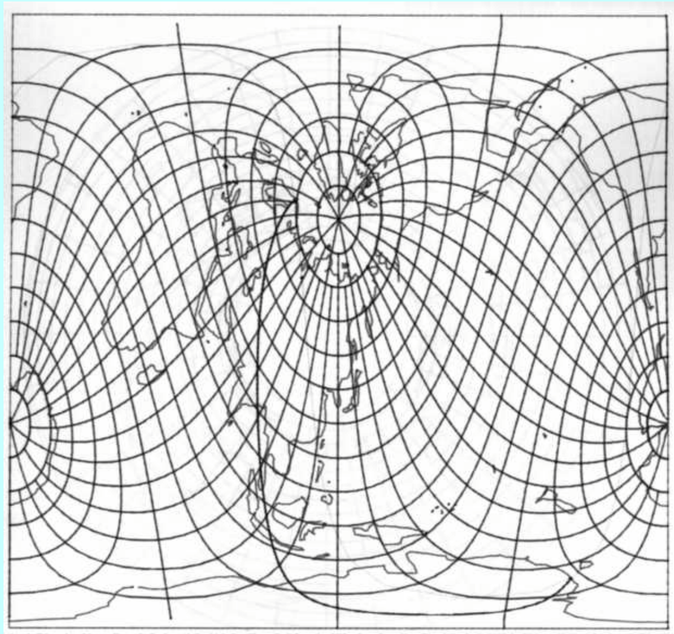
Hammer

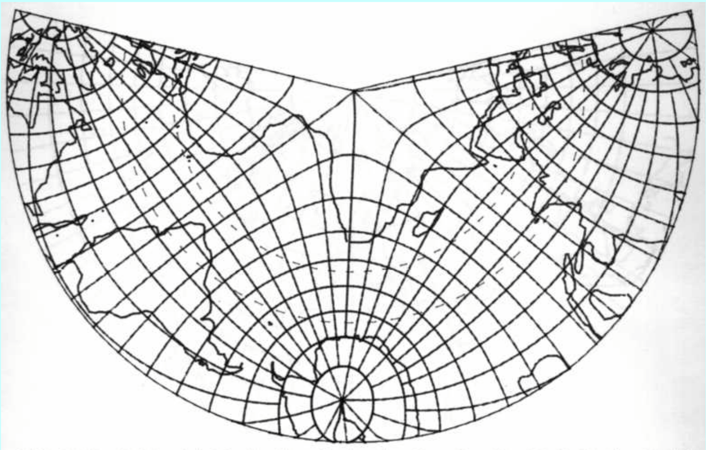


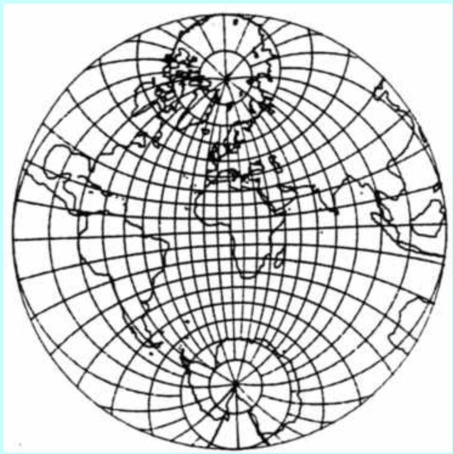
Winkel

Querachsige Entwürfe









Mischkarten und umbezifferte Entwürfe