# Reachability Analysis of Multithreaded Software with Asynchronous Communication

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**Abstract.** We introduce *asynchronous dynamic pushdown networks* (ADPN), a new model for multithreaded programs in which pushdown systems communicate via shared memory. ADPN generalizes both CPS (concurrent pushdown systems) [7] and DPN (dynamic pushdown networks) [5]. We show that ADPN exhibit several advantages as a program model. Since the reachability problem for ADPN is undecidable even in the case without dynamic creation of processes, we address the *bounded* reachability problem [7], which considers only those computation sequences where the (index of the) thread accessing the shared memory is changed at most a fixed given number of times. We provide efficient algorithms for both forward and backward reachability analysis. The algorithms are based on automata techniques for symbolic representation of sets of configurations.

## 1 Introduction

In recent years a number of formalisms have been proposed for modelling and analyzing procedural multithreaded programs. A well-known result states that, if recursion is allowed, checking assertions for these programs is undecidable, even if all variables are boolean (see for instance [8]).

Due to this undecidability result, approximate analysis techniques have been considered. While [3, 4] deal with overapproximations of the set of reachable states, [7] presents the first nontrivial technique to compute underapproximations. In this paper we build on the ideas of [7], which we now describe in some more detail. Qadeer and Rehof introduce *concurrent pushdown systems* (CPS) as a model of multithreaded programs. A CPS is a set of stacks with a global finite control; at each step, the CPS reads the current control state and the topmost symbol of (exactly) one of the stacks, can change the control state and replace the stack symbol by a word, like in a pushdown automaton. A *dynamic* CPS (or DCPS) can also create a new stack as the result of a transition. Each stack of a CPS corresponds to a thread. Communication between threads is modelled through the common set of global control states. A *context* is defined as a computation in which all transitions act *on the same stack*. In [7] it is shown how to compute, given a fixed number $k$, the set of states that can be reached by $k$*-bounded* computations, i.e., by computations consisting of the concatenation of at most $k$ contexts. Obviously, this set constitutes an underapproximation of the set of all reachable states.

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In this paper, we show that with the help of a refined model it is possible to generalize and improve the results of [7] in a number of ways. We propose a generalization of CPS called *asynchronous pushdown networks* (APN); we also introduce the dynamic version of the model, called ADPN. Loosely speaking, the stacks of an APN have an additional set of local control states, different from the common global finite control; transitions are either local (dependent only on the local control), or global (depending on both the global and local control states). We also propose a new, more liberal, definition of context: a context is now a computation in which all *global* transitions act on the same stack, possibly interspersed with local transitions acting on arbitrary stacks.

In the first part of the paper (Section 2) we observe that, while the APN and CPS formalisms are equally expressive, APN can model programs more succinctly than CPS. In the dynamic case we show that, while ADPN can naturally model value passing from a called procedure to its caller, DCPS cannot.

In the second part of the paper (Section 3), we study the forward and backward $k$-bounded reachability problem for APN. Comparing [7], we propose a more general and asymptotically faster algorithm for forward reachability. We introduce a backward reachability algorithm as well.

In the third part of the paper (Sections 4 and 5), we consider the $k$-reachability problem for the ADPN model. We show that, due to the more liberal notion of context, the set of configurations of an ADPN reachable by $k$-bounded computations may be non-regular, contrary to the case of DCPSs. Using results of [5], we show that the set is always context-free and provide an algorithm to compute a context-free grammar that generates it. We then observe that the set of backwards $k$-bounded reachable configurations is regular, and, relying on results from [6], provide an efficient algorithm to compute it.

## 2 The model

### 2.1 Asynchronous dynamic pushdown networks

An *asynchronous dynamic pushdown network (ADPN)* is a tuple $\mathcal{N} = (G, P, \Gamma, \Delta_l, \Delta_g)$, where $G$ is a finite set of *global states*, $P$ is a finite set of *local states*, $\Gamma$ is a finite *stack alphabet*, and

- $\Delta_l$ is a finite set of *local rules* of the form $p\gamma \hookrightarrow p_1 w_1$ or $p\gamma \hookrightarrow p_1 w_1 \rhd p_2 w_2$, where $p, p_1, p_2 \in P$, $\gamma \in \Gamma$, and $w_1, w_2 \in \Gamma^*$.
- $\Delta_g$ is a finite set of *global rules* of the form $(g, p\gamma) \hookrightarrow (g', p_1 w_1)$ or $(g, p\gamma) \hookrightarrow (g', p_1 w_1) \rhd p_2 w_2$, where $g, g' \in G$, $p, p_1, p_2 \in P$, $\gamma \in \Gamma$, and $w_1, w_2 \in \Gamma^*$.

The rules with a suffix of the form $\rhd p_2 w_2$ are called *dynamic*. A *configuration* of an ADPN is a pair $(g, \alpha) \in G \times (P\Gamma^*)^+$ of a global state $g$ and a word $\alpha = p_1 w_1 p_2 w_2 \dots p_n w_n$, where each subword $p_i w_i \in P\Gamma^*$ represents a configuration of (a pushdown corresponding to) one *component*. A word $p_i w_i$ is called *component configuration*. The set of all configurations is denoted by $\mathcal{C}$.

The transition relation $\to \subseteq \mathcal{C} \times \mathcal{C}$ is defined as follows: $(g, u) \to (g', v)$ if there is

- $p\gamma \hookrightarrow p_1 w_1$ in $\Delta_l$ such that $u = u_1 p\gamma u_2$, $v = u_1 p_1 w_1 u_2$, and $g = g'$, or

– $p\gamma \hookrightarrow p_1w_1 \rhd p_2w_2$ in $\Delta_l$ such that $u = u_1p\gamma u_2$, $v = u_1p_2w_2p_1w_1u_2$, and $g = g'$, or
– $(g, p\gamma) \hookrightarrow (g', p_1w_1)$ in $\Delta_g$ such that $u = u_1p\gamma u_2$ and $v = u_1p_1w_1u_2$, or
– $(g, p\gamma) \hookrightarrow (g', p_1w_1) \rhd p_2w_2$ in $\Delta_g$ such that $u = u_1p\gamma u_2$ and $v = u_1p_2w_2p_1w_1u_2$,

where $u_1 \in (P\Gamma^*)^*$ and $u_2 \in \Gamma^*(P\Gamma^*)^*$. We say that the transition has been performed by the component whose local state changes from $p$ to $p_1$. The transitions generated by global and local rules are called *global* and *local transitions* respectively. A dynamic rule creates a new component starting in component configuration $p_2w_2$.

## 2.2 Subclasses of ADPNs

ADPNs are an extension of several other models. An ADPN with only global states and global rules is a *dynamic concurrent pushdown systems* (DCPS). Formally, a DCPS is an ADPN $(G, P, \Gamma, \Delta_l, \Delta_g)$ satisfying $|P| = 1$ and $\Delta_l = \emptyset$. The DCPS model is studied in [7]. The subclasses of ADPN and DCPS without dynamic rules are called APN and CPS, respectively. Notice that in an APN or CPS all configurations reachable from an initial configuration have the same number of components. Finally, both APNs and CPSs are extensions of pushdown systems (PDS). Formally, a PDS is a CPS in which the initial configuration only has one component.

An ADPN without global variables or global rules is called a DPN. DPNs have been introduced and studied in [5]. Notice that in a DPN there is no communication between different threads.

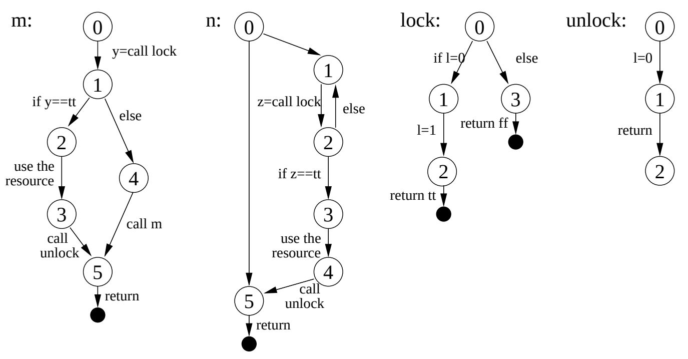
## 2.3 Reachability and bounded reachability

Given an ADPN $\mathcal{N}$ and a set $S \subseteq \mathcal{C}$, we denote by $post^*_{\mathcal{N}}(S)$ and $pre^*_{\mathcal{N}}(S)$ the sets of forward and backward reachable configurations from $S$. The *forward* and *backward reachability problem* consists of, given sets $I$ and $F$ of initial and final configurations, determining if $post^*_{\mathcal{N}}(I) \cap F = \emptyset$ or $pre^*_{\mathcal{N}}(F) \cap I = \emptyset$, respectively. Both problems are undecidable, even when $I$ and $F$ are singletons. This is a consequence of the fact that APNs (even without dynamic rules) are Turing powerful. For instance, it is straightforward to encode a 2-counter Minsky machine into an APN.

Following [7], we define a notion of bounded reachability. A *context* is a transition sequence where all global transitions are performed by the same component. We say that this component *controls* the context. Notice that within a context local transitions can be performed by arbitrary components. For $k \geq 1$, a sequence of transitions is *k-bounded* if it is a concatenation of at most $k$ contexts. We denote by $post^*_{k,\mathcal{N}}(S)$ the set of all configurations reachable from $S$ by $k$-bounded sequences. By analogy, $pre^*_{k,\mathcal{N}}(S)$ denotes the set of all configurations from which a configuration from $S$ is reachable by a $k$-bounded sequence. We talk about *forward* and *backward k-bounded reachability*, respectively. Further, by $post^*_{0,\mathcal{N}}(S)$ and $pre^*_{0,\mathcal{N}}(S)$ we denote the sets of configurations that are forward and backward reachable only by local transitions, respectively.

## 2.4 APN as program model

The following example illustrates how to model programs with APNs (for simplicity, we omit thread creation here). We consider a program with procedures $m, n, lock, unlock$



$\bullet[m_0,b] \hookrightarrow \bullet lock_0[m_1,b]$
$tt[m_1,b] \hookrightarrow \bullet[m_2,tt]$
$ff[m_1,b] \hookrightarrow \bullet[m_4,ff]$
$\bullet[m_2,b] \hookrightarrow \bullet[m_3,b]$
$\bullet[m_3,b] \hookrightarrow \bullet unlock_0[m_5,b]$
$\bullet[m_4,b] \hookrightarrow \bullet[m_0,-][m_5,b]$
$\bullet[m_5,b] \hookrightarrow \bullet\varepsilon$

where $b$ ranges over $\{-,tt,ff\}$

$\bullet[n_0,b] \hookrightarrow \bullet[n_1,b]$
$\bullet[n_0,b] \hookrightarrow \bullet[n_5,b]$
$\bullet[n_1,b] \hookrightarrow \bullet lock_0[n_2,b]$
$tt[n_2,b] \hookrightarrow \bullet[n_3,tt]$
$ff[n_2,b] \hookrightarrow \bullet[n_1,ff]$
$\bullet[n_3,b] \hookrightarrow \bullet[n_4,b]$
$\bullet[n_4,b] \hookrightarrow \bullet unlock_0[n_5,b]$
$\bullet[n_5,b] \hookrightarrow \bullet\varepsilon$

■ $(0,\bullet lock_0) \hookrightarrow (0,\bullet lock_1)$
■ $(1,\bullet lock_0) \hookrightarrow (1,\bullet lock_3)$
■ $(0,\bullet lock_1) \hookrightarrow (1,\bullet lock_2)$
■ $(1,\bullet lock_1) \hookrightarrow (1,\bullet lock_2)$
$\bullet lock_2 \hookrightarrow tt\varepsilon$
$\bullet lock_3 \hookrightarrow ff\varepsilon$
■ $(0,\bullet unlock_0) \hookrightarrow (0,\bullet unlock_1)$
■ $(1,\bullet unlock_0) \hookrightarrow (0,\bullet unlock_1)$
$\bullet unlock_1 \hookrightarrow \bullet\varepsilon$

**Fig. 1.** A program with four procedures and two threads.

described by the flow graphs of Figure 1; $y$ and $z$ are local variables of the procedures $m$ and $n$, respectively, and can take the values undefined $(-)$, true $(tt)$, or false $(ff)$. The procedures $m$ and $n$ call procedures *lock* and *unlock* to get exclusive access to a shared resource. The *lock* action is nonblocking; it returns true if it succeeds to lock the resource, false otherwise. The variable $l$ occurring in the procedures *lock* and *unlock* is global and ranges over $\{0,1\}$. The system consists of two concurrent threads, one starting with the execution of $m$, the other with the execution of $n$.

We model this program by the APN $\mathcal{N} = (G,P,\Gamma,\Delta_l,\Delta_g)$ as follows: Global states model the value of the global variable $l$, i.e. $G = \{0,1\}$. Local states are used to pass a potential return value from a callee back to the caller: The callee stores the value in the local state of the thread, from where it is read by the caller.[3] As a procedure cannot return the undefined value $(-)$, we set $P = \{tt,ff,\bullet\}$, where $tt$ and $ff$ are used to return the corresponding values, and $\bullet$ is used elsewhere. The set $\Gamma$ of stack symbols contains all program locations ($p_l$ denotes the symbol for location $l$ of procedure $p$), together with the actual values of the local variables for procedures $m,n$. The local and global rules corresponding to each procedure are given directly in the figure; global rules (marked with ■) correspond to transitions dealing with the global variable $l$.

The techniques developed in the next sections can show that the program does not satisfy its basic specification: exclusive access to the resource. More precisely, they

[3] In general, local states can be also used to hold values of variables that are global to a thread (if such a variable type is supported in the modeled system).

show that the program can reach a configuration of the form $(0, \bullet[m_2, b]w_1 \bullet [n3, b']w_2)$ from the initial configuration $(0, \bullet[m_0, -] \bullet [n_0, -])$, and in fact within 3 contexts.

### 2.5 A(D)PN versus (D)CPS

As we have seen, local states are used to model value-passing from a callee to its caller. In the CPS model there is no notion of local state of a thread, and so value passing must be simulated through a global variable. Clearly, this amounts to simulating an APN by a CPS. We show that this is possible, but involves a blow-up in size. Moreover, the translation has to fix the number $n$ of components that the CPS can work upon. Let $\mathcal{N} = (G, P, \Gamma, \Delta_l, \Delta_g)$ be an APN. We construct a CPS $\mathcal{N}' = (G', \Gamma', \Delta'_g)$ such that the configuration graphs of $\mathcal{N}$ and $\mathcal{N}'$, defined in the usual way, are isomorphic. We take $G' = G \times P^n$, $\Gamma' = \Gamma \times \{1, \ldots, n\}$, and add to $\Delta'_g$ rules

$$((g_1, p_1, \ldots, p_{i-1}, p, p_{i+1}, \ldots, p_n), q(\gamma, i)) \hookrightarrow ((g_2, p_1, \ldots, p_{i-1}, p', p_{i+1}, \ldots, p_n), q[w, i])$$

for every $(g_1, p\gamma) \hookrightarrow (g_2, p'w)$ in $\Delta_g$, $1 \leq i \leq n$, $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n \in P$, and rules

$$((g, p_1, \ldots, p_{i-1}, p, p_{i+1}, \ldots, p_n), q(\gamma, i)) \hookrightarrow ((g, p_1, \ldots, p_{i-1}, p', p_{i+1}, \ldots, p_n), q[w, i])$$

for every $p\gamma \hookrightarrow p'w$ in $\Delta_l$, $g \in G$, $1 \leq i \leq n$, and $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n \in P$. Here, $q$ is the only local state of $\mathcal{N}'$. Further, for $w = w_1 w_2 \ldots w_m$, $[w, i]$ stands for $(w_1, i)(w_2, i) \ldots (w_m, i)$. Observe that the size of $\mathcal{N}'$ may be larger than that of $\mathcal{N}$ by a factor of $n \cdot |G| \cdot |P|^{n-1}$.

Observe also that the transformation APN $\rightarrow$ CPS cannot be naturally extended to a transformation ADPN $\rightarrow$ DCPS. The straightforward idea of taking $G \times P^*$ as set of global states does not work, and not only because this set is infinite, but also because in order to simulate a change of local state a stack has to know its position in the current state $(g, p_1 p_2 \ldots p_n)$, which now changes as the computation proceeds because of thread creation. Currently we do not know if an ADPN can be translated into an equivalent DCPS, and we do not see any elegant way of modelling value-passing and thread creation in the DCPS formalism.

We finish with an advantage of our more liberal notion of context. In a $k$-bounded computation, at most $k$ components can execute global transitions, and this has the following consequence when comparing ADPN and DCPS: While a $k$-bounded computation of a DCPS can create an arbitrary number of components, at most $k$ of them can execute a transition at all. For ADPN the constraint is weaker: arbitrarily many processes can execute transitions, but at most $k$ of them can execute global transitions. So an algorithm for exploring $k$-bounded computations of ADPN searches 'deeper' as the same algorithm for DCPS.

## 3 Reachability analysis for APN

We now consider $k$-bounded reachability for the APN model, i.e. the restriction of ADPN to non-dynamic rules. Let us fix an APN $\mathcal{N} = (G, P, \Gamma, \Delta_l, \Delta_g)$ and $k \in \mathbb{N}$ for the rest of this section. We investigate the case where the initial or final configurations are given by so-called aggregates:

**Definition 1.** *An* aggregate *is a tuple $M = (g, C_1, \dots, C_n)$, where $g \in G$, $n \geq 1$ is the number of concurrent processes, and $C_1, \dots, C_n \subseteq P \times \Gamma^*$ are* regular *sets of component configurations. $M$ is used to denote the set $\{g\} \times (C_1 . \cdots . C_n)$, where . is the concatenation of the component configurations.*

We now fix an aggregate $M = (g, C_1, \dots, C_n)$ for the rest of the section, and we will present solutions for computing $post^*_{k,\mathcal{N}}(M)$ as well as $pre^*_{k,\mathcal{N}}(M)$.

For the CPS model, $k$-bounded reachability was considered in [7]. The algorithms presented in this section follow the same general idea as the solutions in [7] (but applied to APN). Moreover, the new solution has these benefits:

– Our algorithm avoids repeating partial computations of reachable component configurations. Even if we consider only CPSs, the algorithm runs asymptotically faster than the one presented in [7].
– The APN model distinguishes between local and global states, and our algorithm exploits this difference. Therefore, it is faster than a translation of a given APN to CPS (see Section 2.5) followed by the application of an algorithm for CPS.
– Some details in our algorithm are different from [7] and would lead to time and memory savings in an implementation. These are discussed in Section 3.3.
– We provide algorithms for both forward and backwards reachability, whereas [7] only covered forward reachability. The two algorithms are fairly similar – in fact we will present them as one algorithm – but their complexity analysis is a little more involved. The algorithm makes use of a procedure called CLOSURE, which stands for the *post*$^*$ or *pre*$^*$ procedure on PDSs [6] in case of forward and backwards reachability, respectively.

## 3.1 Reordering of transitions

Our algorithms are based on the following observation: Let $c$ be a configuration reachable from $M = (g, C_1, \dots, C_n)$ by a $k$-bounded computation, and let $\sigma$ be this computation. Then the transitions in $\sigma$ can be rearranged to another $k$-bounded computation $\sigma'$ that also leads from $M$ to $c$. Moreover, $\sigma'$ can be partitioned into $n+k$ phases, where in each phase all rules are applied to the same component:

– In the $i$-th phase, $1 \leq i \leq n$, component $i$ executes all its *local* steps in $\sigma$ up to, but not including, its first global step (or all steps, if it never executes a global rule).
– In the $n+i$-th phase, $1 \leq i \leq k$, the component controlling the $i$-th context executes the first *global* step of the $i$-th context in $\sigma$, followed by all its *global* and *local* steps up to, but not including, the first global step in the next context controlled by the same component (all its remaining steps, if it does not control any more contexts).

Notice that this rearrangement only requires to swap the ordering of local transitions of some component with local or global transitions of other components; but as the application of a local rule does not depend on the global state, these reorderings do not alter the final configuration of the computation.

## 3.2 Reduction to PDS

We now show that all $n+k$ phases reduce to reachability problems on PDS. In the following, $\textsc{Closure}_{\mathcal{P}}(C)$ denotes the set $post^*_{\mathcal{P}}(C)$ or $pre^*_{\mathcal{P}}(C)$, depending on whether forward or backward reachability is of interest.

– Let $\mathcal{P}^1_{\mathcal{N}} := (P,\Gamma,\Delta_l)$, i.e. $\mathcal{P}^1_{\mathcal{N}}$ simulates the local moves of $\mathcal{N}$. Thus, the results of the first $n$ phases are obtained by $\textsc{Closure}_{\mathcal{P}^1_{\mathcal{N}}}(C_i)$ for $i = 1,\dots,n$.
– For the remaining phases, we create a PDS in which the global and local states are merged. Let $\mathcal{P}^2_{\mathcal{N}} = (G \times P,\Gamma,\Delta')$, where $\Delta'$ contains all $(g_1,p_1)\gamma \hookrightarrow (g_2,p_2)w$ such that either $(g_1,p_1\gamma) \hookrightarrow (g_2,p_2w)$ in $\Delta_g$, or $p_1\gamma \hookrightarrow p_2w$ in $\Delta_l$ and $g_1 = g_2$. Thus, $\mathcal{P}^2_{\mathcal{N}}$ computes the possible operations of one component in a single context. More precisely, we define $\textsc{Lift}(g,C) := \{\,((g,p),w) \mid (p,w) \in C\}$ and $\textsc{Restrict}(C,g) := \{\,(p,w) \mid ((g,p),w) \in C\}$. Now, if a component starts a context in global state $g$ and with component configurations $C$, the reachable configurations within this context that end in global state $g'$ are $\textsc{Restrict}(\textsc{Closure}_{\mathcal{P}^2_{\mathcal{N}}}(\textsc{Lift}(g,C)),g')$.

Recall that the initial sets $C_1,\dots,C_n$ are regular and can be represented by finite automata. Regular sets are closed under the $\textsc{Closure}$ operation, and algorithms for these have been provided in [6]. It is easy to see that $\textsc{Lift}$ and $\textsc{Restrict}$ can also be implemented as operations on finite automata.

## 3.3 The algorithm

Figure 2 shows our algorithm, which directly implements the ideas outlined before. Line 2 computes the local phases $1,\dots,n$ of the computations, whereas the lines from line 3 onwards implement phases $n+1,\dots,n+k$. Essentially, the algorithm explores a 'tree' of depth $k$, where each node corresponds to an aggregate, and its successors are the aggregates reachable by executing one context. Each iteration of the while loop picks an aggregate and computes its successors. As hinted at before, the operations on the sets of component configurations are carried out by operations on finite automata. The algorithm uses the following data structures:

*todo* is a list with information on those aggregates whose successors still need to be computed. The first part of each entry in *todo* indicates the depth of the aggregate in the tree, the second is the index of the component that has controlled the previous context; the rest is the aggregate itself.

*aut* is a hash table. An entry $aut[g,B]$ remembers the result of applying the closure on $\textsc{Lift}(g,B)$. The motivation for this table is that, for a pair $(g,B)$, the computation of $\textsc{Closure}_{\mathcal{P}^2_{\mathcal{N}}}(\textsc{Lift}(g,B))$ may be required in multiple branches of the 'tree'; therefore we would like to reuse the result. Notice that actually hashing over (an automaton accepting) the language $B$ could be very time consuming. In order to achieve the desired time-saving effect, it suffices to approximate this effect, e.g. by giving a unique identifier to each automaton that arises from an application of $\textsc{Closure}$.

*reachable* collects the aggregates that represent reachable configurations.

**Input:** An APN $\mathcal{N}$, an aggregate $M = (g, C_1, \ldots, C_n)$, and $k \in \mathbb{N}$
**Output:** The set $post^*_{k,\mathcal{N}}(M)$ (or $pre^*_{k,\mathcal{N}}(M)$) given by union of the aggregates in *reachable*.

```
1   $reachable \leftarrow \emptyset$;
2   $todo \leftarrow \{(0, 0, g, \textsc{Closure}_{\mathcal{P}^1_{\mathcal{N}}}(C_1), \ldots, \textsc{Closure}_{\mathcal{P}^1_{\mathcal{N}}}(C_n))\}$;
3   while $todo \neq \emptyset$ do
4     pop $(level, last, g, B_1, \ldots, B_n)$ with minimal $level$ from $todo$;
5     if $level = k$ then
6       $reachable \leftarrow reachable \cup \{(g, B_1, \ldots, B_n)\}$;
7     else
8       for all $i = 1, \ldots, n$ such that $i \neq last$ do
9         if $aut[g, B_i]$ undefined then
10          $aut[g, B_i] \leftarrow \textsc{Closure}_{\mathcal{P}^2_{\mathcal{N}}}(\textsc{Lift}(g, B_i))$;
11        for all $g' \in G$ do
12          $todo \leftarrow todo \cup \{(level+1, i, g', B_1, \ldots, B_{i-1}, \textsc{Restrict}(aut[g, B_i], g'), B_{i+1}, \ldots, B_n)\}$;
```

**Fig. 2.** Algorithm computing $k$-bounded reachability on APN.

The basic idea of exploring a tree of depth $k$ is similar to the CPS algorithm in [7]. However, the algorithm in Figure 2 also contains some improvements:

– When adding a new item to *todo*, the algorithm reuses all previous local automata except for $B_i$ (unlike [7], where all $n$ automata are changed in every step). This makes the algorithm more memory-efficient, because the automata that have not changed from one context to another can be shared.
– Using *aut* allows to reuse results of computations made in other parts of the tree.
– A trivial improvement is that no component is allowed to execute two contexts in a row (the second context would yield nothing new due to closure properties).
– Another simple, but important optimization (not shown) is that line 11 should only be executed for those global states $g'$ such that $aut[g, B_i]$ accepts at least one configuration of the form $\langle g', w\rangle$ for some $w \in \Gamma^*$.

## 3.4 Complexity analysis

We now state the complexity of our algorithm for both directions. The proofs can be found in [2]. Let $A_1, \ldots, A_n$ be automata representing $C_1, \ldots, C_n$.

**Theorem 1.** *Let $M = (g, C_1, \ldots, C_n)$ be an aggregate of an APN $\mathcal{N} = (G, P, \Gamma, \Delta_l, \Delta_g)$ and let $k \in \mathbb{N}$ be a number. Then there exist aggregates $M_0, \ldots, M_m$ such that $post^*_{k,\mathcal{N}}(M)$ (or $pre^*_{k,\mathcal{N}}(M)$, resp.) has the form $M_0 \cup M_1 \cup \ldots \cup M_m$ and all these aggregates are effectively computable. Moreover,*

*(a) computing $post^*_{k,\mathcal{N}}(M)$ takes $O(n^k \cdot |G|^k + n \cdot |G|^k \cdot |P| \cdot (d + |\Delta| \cdot k \cdot q + |\Delta|^2 \cdot k^2))$ time, where $|\Delta| = |G| \cdot |\Delta_l| + |\Delta_g|$ and $q, d$ are the largest numbers of non-initial states and transitions leading out of non-initial states in $A_1, \ldots, A_n$, respectively;*

*(b)* $pre^*_{k,\mathcal{N}}(M)$ *can be computed in time* $O(n^k \cdot |G|^k + n \cdot |G|^{k-1} \cdot (q + k \cdot |P| \cdot |G|)^2 \cdot |\Delta|)$ *where* $|\Delta| = |G| \cdot |\Delta_l| + |\Delta_g|$ *and q is the maximal number of states in* $A_1, \dots, A_n$.

Note that the complexity given for $k$-bounded forward CPS reachability in [7] has (among others) the factors $k^3$ and $|G|^{k+5}$. Seeing as APNs are an extension of CPSs, Theorem 1 provides a better upper bound for $k$-bounded reachability even on CPSs.

## 4 Forward reachability analysis of ADPN

Even in the DPN case, the $post^*$ image of a regular set of configurations is not always regular [5]. However, it can be shown that this image is always context-free, and [5] provides a construction that, given a DPN and an initial configuration $p_0\gamma_0$, computes a context-free grammar $\mathcal{G}$ such that $L(\mathcal{G}) = post^*(p_0\gamma_0)$.

In this paper we show how to compute $post^*_{k,\mathcal{N}}(c_0)$ for an ADPN $\mathcal{N}$, a configuration $c_0 = (g_0, p_0\gamma_0)$ and an arbitrary $k \geq 0$. (The algorithm can be extended from one configuration $c_0$ to a regular set of configurations.) The key of the result is a construction which, given a sequence $\sigma = g_0 \dots g_{k-1}$ of global states of $\mathcal{N}$, constructs a DPN $\mathcal{N}_\sigma$, a configuration $c$, a regular set $S$, and a regular transduction $\pi$ (as we shall see, $S$, $c$, and $\pi$ are independent from $\sigma$) such that $post^*_{k,\mathcal{N}}(c_0) = \pi(S \cap \bigcup_{\sigma \in G^k} post^*_{\mathcal{N}_\sigma}(c))$. By the result of [5], the sets $post^*_{\mathcal{N}_\sigma}(c)$ are effectively context-free, and so $post^*_{k,\mathcal{N}}(c_0)$ is effectively context-free as well.

Informally, given $\sigma = g_0 \dots g_{k-1}$ the DPN $\mathcal{N}_\sigma$ is able to simulate those execution sequences of $\mathcal{N}$ in which, for every $1 \leq i < k$, the $i$-th context-switch occurs at a configuration of $\mathcal{N}$ with global state $g_i$. During the simulation, each pushdown component of $\mathcal{N}_\sigma$ maintains a guess about the index of the current context. (Notice that, due to the lack of communication between components of a DPN, a component cannot know how many context-switches have occurred). The component can at any point increase its guess, but cannot decrease it. A wrong guess leads to an unfaithful simulation (see below how to 'filter them away'). Moreover, the component can at any point decide to control the current context (more precisely, the context it guesses is the current one). In such a case, the current global state is mantained as a part of the corresponding local state. Since components cannot communicate, this may lead to an unfaithful simulation, where zero, two or more different components claim to control the same context.

The problem of the unfaithful simulations is solved with the help of the set $S$ and the homomorphism $\pi$. We define $\mathcal{N}_\sigma$ so that if a component completes the simulation of a context it claims to have controlled, then it must create an inactive 'marker' (a new component that can do nothing) witnessing this claim. At the end of the simulation we can inspect the inactive markers, and check if every context was indeed controlled by one and at most one component. If this is so, the simulation is faithful, otherwise it is unfaithful. The set $S$ is the set of configurations where every marker appears exactly once, and so intersection with $S$ 'filters out' all the configurations reached by faithful simulations. The transduction $\pi$ is used to 'clean up' the configurations so obtained by disposing of the markers and other auxiliary symbols used along the simulation, and to move the global state (stored in the local state of the process controlling the last context) to the front of the configuration.

For details of the construction of $\mathcal{N}_\sigma$ we refer to [2]. The construction gives rise to the following theorem:

**Theorem 2.** *Let $\mathcal{N} = (G,P,\Gamma,\Delta_l,\Delta_g)$ be an ADPN and let $c_0 = (g_0, p_0\gamma_0)$ be a configuration of $\mathcal{N}$. The set $post^*_{k,\mathcal{N}}(c_0)$ is context-free. A context-free grammar generating it can be constructed in time $O(k^3 \cdot |G|^{k+3} \cdot |P|^3 \cdot (|\Delta_l| + |\Delta_g|))$*

## 5 Backward reachability analysis of ADPN

We consider here the problem of constructing the $pre^*_k$ images of a regular set of configurations, under the assumption of at most $k$ contexts. We provide a reduction of this problem to the problem of computing $pre^*$ images in the case of DPNs (or in other words to the problem of computing $pre^*_1$ images), and we provide and efficient algorithm for solving the latter problem. This algorithm improves the complexity of the basic saturation-based procedure proposed in [5] for symbolic backward reachability analysis of DPN.

### 5.1 Regular symbolic representations

Our algorithms use a class of automata-based representations for regular sets of configurations (mass configurations) which have been introduced in [5] for DPN analysis. These representations are finite-state automata in a *special form* defined below.

Let $\mathcal{N} = (G,P,\Gamma,\Delta_l,\Delta_g)$ be an ADPN. Then, a finite-state automaton $A = (Q,\Sigma,\delta,q_0,F)$ is called *$\mathcal{N}$-automaton* if and only if it satisfies the following conditions:

- $\Sigma = P \cup \Gamma$,
- $Q$ can be partitioned into three mutually disjoint subsets $Q_0, Q_1, Q_2$ such that for all $q \in Q_0, p \in P$ there exists a unique state $q_p \in Q_1$,
- transition relation $\delta$ can be partitioned into three disjoint relations $\delta_0, \delta_1, \delta_2$ such that $\delta_0 = \{(q,p,q_p) \mid q \in Q_0, p \in P, q_p \in Q_1\}$, $\delta_1 \subseteq (Q_1 \cup Q_2) \times \Gamma \times Q_2$, and $\delta_2 \subseteq (Q_1 \cup Q_2) \times \{\varepsilon\} \times Q_0$,
- $q_0 \in Q_0$, and $F \subseteq Q_1 \cup Q_2$.

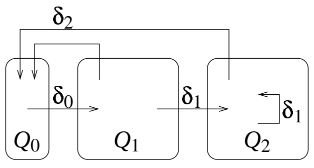


**Fig. 3.** An automaton in special form.

An automaton in the above special form is schematically depicted in Figure 3. Notice that $\mathcal{N}$-automata recognize languages which are regular subsets of $(P\Gamma^*)^+$. It is easy to see that, conversely, every finite-state automaton over the alphabet $\Sigma = P \cup \Gamma$ recognizing a language included in $(P\Gamma^*)^+$ can be transformed into a language equivalent $\mathcal{N}$-automaton. Notice also that this definition depends obviously on the model $\mathcal{N}$ under consideration, but only on his set of control states $P$ and his stack alphabet $\Gamma$ and not on the fact whether global variables and rules are considered.

Following the common habit, we write $q \xrightarrow{a}_\delta q'$ meaning $(q,a,q') \in \delta$. We also extend this notation to finite words in standard way: for every $q, q' \in Q$, $a \in \Sigma$ and $u \in \Sigma^*$ we set $q \xrightarrow{\varepsilon}_\delta q$ and $q \xrightarrow{au}_\delta q'$ iff there is $q'' \in Q$ such that $q \xrightarrow{a}_\delta q''$ and $q'' \xrightarrow{u}_\delta q'$.

### 5.2 Computing $pre^*$ images for DPN

Let $\mathcal{N} = (P,\Gamma,\Delta)$ be a DPN and $A = (Q,\Sigma,\delta,q_0,F)$ be an $\mathcal{N}$-automaton. We describe a simple procedure proposed in [5] for computing a finite-state automaton $A_{pre^*}$ satisfying $L(A_{pre^*}) = pre^*_{\mathcal{N}}(L(A))$. The automaton is defined as $A_{pre^*} = (Q,\Sigma,\delta',q_0,F)$, where $\delta'$ is the smallest relation $\delta' \supseteq \delta$ satisfying the following two conditions.

– If $p\gamma \hookrightarrow p_1w_1 \in \Delta$ and $q \xrightarrow{p_1w_1}_{\delta'} q'$ for $q,q' \in Q$ then $(q_p,\gamma,q') \in \delta'$.
– If $p\gamma \hookrightarrow p_1w_1 \rhd p_2w_2 \in \Delta$ and $q \xrightarrow{p_2w_2p_1w_1}_{\delta'} q'$ for $q,q' \in Q$ then $(q_p,\gamma,q') \in \delta'$.

The construction of the automaton $A_{pre^*}$ terminates since it corresponds to adding iteratively new transitions to the original automaton $A$ without modifying the number of its states. The construction can be proved to be sound and complete [5].

It can be seen that this construction is polynomial but a naive implementation of it can be of a prohibitive cost, similarly to the basic algorithm of [1] for pushdown systems with respect to its efficient implementation of [6]. Following the principles used in [6], we define an efficient algorithm implementing the saturation-based procedure above (see [2]). We have the following result:

**Theorem 3.** *Given a DPN $\mathcal{N} = (P,\Gamma,\Delta)$ and an $\mathcal{N}$-automaton $A = (Q,\Sigma,\delta,q_0,F)$, it is possible to construct in $O(|Q|^3 \cdot |\Delta|)$ time and $O(|Q|^2 \cdot |\Delta|)$ space an automaton $A_{pre^*}$ such that $L(A_{pre^*}) = pre^*(L(A))$.*

### 5.3 Computing $pre^*_k$ images for ADPN

Let $\mathcal{N} = (G,P,\Gamma,\Delta_l,\Delta_g)$ be an ADPN, and let $k \geq 1$. Roughly speaking, the computation of a $pre^*_{k,\mathcal{N}}$ image is decomposed into $k$ successive steps of $pre^*_{1,\mathcal{N}}$ image computation, each of them consisting basically in a $pre^*$ image computation in a (suitably defined) DPN. To define in more details the construction, we need some notations and definitions. A *mass configuration* is a pair $M = (g,A)$. It represents the set of configurations $(g,u)$ where $u \in L(A)$. Given a mass configuration $M = (g,A)$, let $local(M)$ denote the automaton $A$. We generalize this notation to finite collections of mass configurations by taking the union of their $\mathcal{N}$-automata.

Then, given a mass configuration $(g,A)$, the computation of $pre^*_{k,\mathcal{N}}(g,A)$ is performed as follows: first we compute the set $pre^*_{1,\mathcal{N}}(g,A)$ corresponding to all predecessors of $(g,A)$ without context switch. For every global state $g'$, let $(g',A')$ be the set of all configurations in $pre^*_{1,\mathcal{N}}(g,A)$ having $g'$ as global state. Then, the second step constists in computing the $pre^*_{1,\mathcal{N}}$ images of all the pairs $(g',A')$, for all global states $g'$, and so on. More precisely, given an $\mathcal{N}$-automaton $A$ and a sequence of global states $\sigma \in G^+$, we define inductively the set $\textsc{Reach}_\sigma(A)$:

$$\textsc{Reach}_g(A) = pre^*_{1,\mathcal{N}}(g,A)$$
$$\textsc{Reach}_{g_1g_2\sigma'}(A) = \textsc{Reach}_{g_2\sigma'}(local(\textsc{Reach}_{g_1}(A) \cap (g_2,(P\Gamma^*)^+)))$$

where $g,g_1,g_2 \in G$ and $\sigma' \in G^*$. Then, the following fact holds.

**Lemma 1.** *Given an ADPN $\mathcal{N}$, a global state $g$, an $\mathcal{N}$-automaton $A$, and an integer $k \geq 1$, we have $pre^*_{k,\mathcal{N}}(g,A) = \bigcup_{g_1,\ldots,g_{k-1} \in G^{k-1}} \textsc{Reach}_{gg_1\cdots g_{k-1}}(A)$.*

Therefore, we only have to show how to construct $pre^*_{1,\mathcal{N}}$ images. For that, we can actually use our algorithm of Theorem 3 which allows to perform backward analysis for DPN. Given an $\mathcal{N}$-automaton $A$ and a global state $g$, we proceed as follows:

– we construct an automaton $\widehat{A}$ such that for every word $u$ of component configurations which is accepted by $A$, the automaton $\widehat{A}$ accepts all words arising from $u$ by embedding the global state $g$ into a local state of one of the components. More precisely, $\widehat{A}$ accepts a word $w$ if and only if there is a word $u_1 p u_2 \in L(A)$ such that $u_1 \in (P\Gamma^*)^*$, $p \in P$, $u_2 \in \Gamma^*(P\Gamma^*)^*$, and $w = u_1(g,p)u_2$.
– we transform the sets $\Delta_l$ and $\Delta_g$ into a set of *local rules* $\Delta$ which are applicable to local states (with an embedded global state). The set of obtained rules has a size $O(|G| \cdot |\Delta_l| + |\Delta_g|)$.
– we use the algorithm for DPN of Theorem 3 to build an automaton $\widehat{A}_{pre^*}$.
– then,

$$pre^*_{1,\mathcal{N}}(g,A) = \bigcup_{g' \in G} (g', \{w \in (P\Gamma^*)^+ \;:\; w = upu' \text{ and } \exists u(g',p)u' \in L(\widehat{A}_{pre^*})\}).$$

An automata-based representation for this set can be straightforwardly obtained from $\widehat{A}_{pre^*}$ using intersection and projection. Then we have the following (see [2]).

**Theorem 4.** *Given an ADPN $\mathcal{N} = (G,P,\Gamma,\Delta_l,\Delta_g)$, $k \geq 1$, $g \in G$, and an $\mathcal{N}$-automaton $A = (Q,\Sigma,\delta,q_0,F)$, it is possible to construct a finite-state automata-based representation of the set $pre^*_{k,\mathcal{N}}(g,A)$ in $O(k^4 \cdot |Q|^3 \cdot (|G|^k \cdot |\Delta_l| + |G|^{k-1} \cdot |\Delta_g|))$ time.*

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