# Word Problems for 2-Homogeneous Monoids and Symmetric Logspace

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**Abstract.** We prove that the word problem for every monoid presented by a fixed 2-homogeneous semi-Thue system can be solved in log-space, which generalizes a result of Lipton and Zalcstein for free groups. The uniform word problem for the class of all 2-homogeneous semi-Thue systems is shown to be complete for symmetric log-space.

## 1 Introduction

Word problems for finite semi-Thue systems, or more precisely word problems for monoids presented by finite semi-Thue systems, received a lot of attention in mathematics and theoretical computer science and are still an active field of research. Since the work of Markov [18] and Post [22] it is known that there exists a fixed semi-Thue system with an undecidable word problem. This has motivated the search for classes of semi-Thue systems with decidable word problems and the investigation of the computational complexity of these word problems, see e.g. [9, 7, 16]. In [1] Adjan has investigated a particular class of semi-Thue systems, namely $n$-homogeneous systems, where a semi-Thue system is called $n$-homogeneous if all rules are of the form $s \to \epsilon$, where $s$ is a word of length $n$ and $\epsilon$ is the empty word. Adjan has shown that there exists a fixed 3-homogeneous semi-Thue system with an undecidable word problem and furthermore that every 2-homogeneous semi-Thue system has a decidable word problem. Book [8] has sharpened Adjan's decidability result by proving that the word problem for every 2-homogeneous semi-Thue system can be solved in linear time.

In this paper we will continue the investigation of 2-homogeneous semi-Thue systems. In the first part of the paper we will prove that the word problem for every 2-homogeneous semi-Thue system can be solved in logarithmic space. This result improves Adjan's decidability result in another direction and also generalizes a result of Lipton and Zalcstein [15], namely that the word problem for a finitely generated free group can be solved in logarithmic space. Furthermore our log-space algorithm immediately shows that the word problem for an arbitrary 2-homogeneous semi-Thue system can be solved in DLOGTIME-uniform $\mathrm{NC}^1$ if the word problem for the free group of rank 2 is solvable in DLOGTIME-uniform $\mathrm{NC}^1$. Whether the later holds is one of the major open questions concerning the class DLOGTIME-uniform $\mathrm{NC}^1$. In the second part of this paper we will consider the uniform word problem for 2-homogeneous semi-Thue systems. In this decision problem the 2-homogeneous semi-Thue system is also part of the input. Building on the results from the first part, we will show that the uniform word problem for the class of all 2-homogeneous semi-Thue systems is complete for symmetric log-space. This result is in particular interesting from the viewpoint of computational complexity, since there are quite few natural and nonobvious SL-complete problems in formal language theory, see [2].

## 2 Preliminaries

We assume some familiarity with computational complexity, see e.g. [21], in particular with circuit complexity, see e.g. [27]. L denotes deterministic logarithmic space. SL (symmetric log-space) is the class of all problems that can be solved in log-space on a symmetric (nondeterministic) Turing machine, see [14] for more details. Important results for SL are the closure of SL under log-space bounded Turing reductions, i.e., $\mathrm{SL} = \mathrm{L}^{\mathrm{SL}}$ [19], and the fact that problems in SL can be solved in deterministic space $O(\log(n)^{\frac{4}{3}})$ [3]. A collection of SL-complete problems can be found in [2]. For the definition of DLOGTIME-uniformity and DLOGTIME-reductions see e.g. [10, 5]. DLOGTIME-uniform $\mathrm{NC}^1$, briefly $\mathrm{uNC}^1$, is the class of all languages that can be recognized by a DLOGTIME-uniform family of polynomial-size, logarithmic-depth, fan-in two Boolean circuits. It is well known that $\mathrm{uNC}^1$ corresponds to the class ALOGTIME [24]. An important subclass of $\mathrm{uNC}^1$ is DLOGTIME-uniform $\mathrm{TC}^0$, briefly $\mathrm{uTC}^0$. It is characterized by DLOGTIME-uniform families of constant depth, polynomial-size, unbounded fan-in Boolean circuits with majority-gates. Using the fact that the number of 1s in a word over $\{0, 1\}$ can be calculated in $\mathrm{uTC}^0$ [5], the following result was shown in [4].

**Theorem 1.** *The Dyck-language over 2 bracket pairs is in $uTC^0$.*

By allowing more than one output gate in circuits we can speak of functions that can be calculated in $\mathrm{uTC}^0$. But with this definition only functions $f : \{0, 1\}^* \to \{0, 1\}^*$ that satisfy the requirement that $|f(x)| = |f(y)|$ if $|x| = |y|$ could be computed. In order to overcome this restriction we define for a function $f : \{0, 1\}^* \to \{0, 1\}^*$ the function $\mathrm{pad}(f) : \{0, 1\}^* \to \{0, 1\}^*\{\#\}^*$ by $\mathrm{pad}(f)(x) = y\#^n$, where $f(x) = y$ and $n = \max\{|f(z)| \mid z \in \{0, 1\}^{|x|}\} - |y|$. Then we say that a function $f$ can be calculated in $\mathrm{uTC}^0$ if the function $\mathrm{pad}(f)$ can be calculated by a family of circuits that satisfy the restrictions for $\mathrm{uTC}^0$, where the alphabet $\{0, 1, \#\}$ has to be encoded into the binary alphabet $\{0, 1\}$. Hence we also have a notion of $\mathrm{uTC}^0$ many-one reducibility. More generally we say that a language $A$ is $\mathrm{uTC}^0$-*reducible* to a language $B$ if $A$ can be recognized by a DLOGTIME-uniform family of polynomial-size, constant-depth, unbounded fan-in Boolean circuits containing also majority-gates and oracle-gates for the language $B$. This notion of reducibility is a special case of the $\mathrm{NC}^1$-reducibility of [11]. In particular [11, Proposition 4.1] immediately implies that L is closed under $\mathrm{uTC}^0$-reductions. Moreover also $\mathrm{uNC}^1$ and $\mathrm{uTC}^0$ are closed under $\mathrm{uTC}^0$-reducibility and $\mathrm{uTC}^0$-reducibility is transitive. The following inclusions are known between the classes introduced above: $\mathrm{uTC}^0 \subseteq \mathrm{uNC}^1 \subseteq \mathrm{L} \subseteq \mathrm{SL}$.

For a binary relation $\rightarrow$ on some set we denote by $\xrightarrow{*}$ the reflexive and transitive closure of $\rightarrow$. In the following let $\Sigma$ be a finite alphabet. An *involution* $\overline{\phantom{a}}$ on $\Sigma$ is a function $\overline{\phantom{a}}: \Sigma \rightarrow \Sigma$ such that $\overline{\overline{a}} = a$ for all $a \in \Sigma$. The empty word over $\Sigma$ is denoted by $\epsilon$. Let $s = a_1 a_2 \cdots a_n \in \Sigma^*$ be a word over $\Sigma$, where $a_i \in \Sigma$ for $1 \leq i \leq n$. The *length* of $s$ is $|s| = n$. For $1 \leq i \leq n$ we define $s[i] = a_i$ and for $1 \leq i \leq j \leq n$ we define $s[i, j] = a_i a_{i+1} \cdots a_j$. If $i > j$ we set $s[i, j] = \epsilon$. Every word $s[1, i]$ with $i \geq 1$ is called a *non-empty prefix* of $s$. A *semi-Thue system* $\mathcal{R}$ over $\Sigma$, briefly STS, is a finite set $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$. Its elements are called *rules*. See [13, 6] for a good introduction into the theory of semi-Thue systems. The length $\|\mathcal{R}\|$ of $\mathcal{R}$ is defined by $\|\mathcal{R}\| = \sum_{(s,t)\in\mathcal{R}} |st|$. As usual we write $x \rightarrow_{\mathcal{R}} y$ if there exist $u, v \in \Sigma^*$ and $(s, t) \in \mathcal{R}$ with $x = usv$ and $y = utv$. We write $x \leftrightarrow_{\mathcal{R}} y$ if ($x \rightarrow_{\mathcal{R}} y$ or $y \rightarrow_{\mathcal{R}} x$). The relation $\xleftrightarrow{*}_{\mathcal{R}}$ is a congruence with respect to the concatenation of words, it is called the *Thue-congruence* generated by $\mathcal{R}$. Hence we can define the quotient monoid $\Sigma^*/\xleftrightarrow{*}_{\mathcal{R}}$, which is briefly denoted by $\Sigma^*/\mathcal{R}$. A word $t$ is a $\mathcal{R}$*-normalform* of $s$ if $s \xrightarrow{*}_{\mathcal{R}} t$ and $t$ is $\mathcal{R}$*-irreducible*, i.e., there does not exist a $u$ with $t \rightarrow_{\mathcal{R}} u$. The STS $\mathcal{R}$ is *confluent* if for all $s, t, u \in \Sigma^*$ with ($s \xrightarrow{*}_{\mathcal{R}} t$ and $s \xrightarrow{*}_{\mathcal{R}} u$) there exists a $v$ with ($t \xrightarrow{*}_{\mathcal{R}} v$ and $u \xrightarrow{*}_{\mathcal{R}} v$). It is well-known that $\mathcal{R}$ is confluent if and only if $\mathcal{R}$ is *Church-Rosser*, i.e., for all $s, t \in \Sigma^*$ if $s \xleftrightarrow{*}_{\mathcal{R}} t$ then ($s \xrightarrow{*}_{\mathcal{R}} u$ and $t \xrightarrow{*}_{\mathcal{R}} u$) for some $u \in \Sigma^*$, see [6, p 12]. For a morphism $\phi : \Sigma^* \rightarrow \Gamma^*$ we define the STS $\phi(\mathcal{R}) = \{(\phi(\ell), \phi(r)) \mid (\ell, r) \in \mathcal{R}\}$. Let $n \geq 1$. A STS $\mathcal{R}$ is $n$*-homogeneous* if all rules of $\mathcal{R}$ have the form $(\ell, \epsilon)$ with $|\ell| = n$. An important case of a confluent and 2-homogeneous STS is the STS $\mathcal{S}_n = \{c_i\overline{c}_i \rightarrow \epsilon,\ \overline{c}_i c_i \rightarrow \epsilon \mid 1 \leq i \leq n\}$ over $\Gamma_n = \{c_1, \ldots, c_n, \overline{c}_1, \ldots, \overline{c}_n\}$. The monoid $\Gamma_n^*/\mathcal{S}_n$ is the *free group* $F_n$ of rank $n$.

A decision problem that is of fundamental importance in the theory of semi-Thue systems is the uniform word problem. Let $\mathcal{C}$ be a class of STSs. The *uniform word problem*, briefly UWP, for the class $\mathcal{C}$ is the following decision problem:

INPUT: An $\mathcal{R} \in \mathcal{C}$ (over some alphabet $\Sigma$) and two words $s, t \in \Sigma^*$.

QUESTION: Does $s \xleftrightarrow{*}_{\mathcal{R}} t$ hold?

Here the length of the input is $\|\mathcal{R}\| + |st|$. The UWP for a singleton class $\{\mathcal{R}\}$ is called the *word problem*, briefly WP, for $\mathcal{R}$. In this case we also speak of the word problem for the monoid $\Sigma^*/\mathcal{R}$ and the input size is just $|st|$. In [15] the word problem for a fixed free group was investigated and the following theorem was proven as a special case of a more general result on linear groups.

**Theorem 2.** *The WP for the free group $F_2$ of rank 2 is in L.*

This result immediately implies the following corollary.

**Corollary 1.** *The UWP for the class $\{\mathcal{S}_n \mid n \geq 1\}$ is uTC$^0$-reducible to the WP for $F_2$, and therefore is also in L.*

*Proof.* The group morphism $\varphi_n : F_n \rightarrow F_2$ defined by $c_i \mapsto \overline{c}_1^i c_2 c_1^i$ is injective, see e.g. [17, Proposition 3.1]. Furthermore $\varphi_n(w)$ can be calculated from $w$ and $\mathcal{S}_n$ in uTC$^0$. The second statement of the theorem follows with Theorem 2. □

Finally let us mention the following result, which was shown in [23].

**Theorem 3.** *The WP for the free group $F_2$ of rank 2 is uNC$^1$-hard under DLOGTIME-reductions.*

## 3 The confluent case

In this section we will investigate the UWP for the class of all confluent and 2-homogeneous STSs. For the rest of this section let $\mathcal{R}$ be a confluent and 2-homogeneous semi-Thue system over an alphabet $\Sigma$. It is easy to see that w.l.o.g. we may assume that every symbol in $\Sigma$ appears in some rule of $\mathcal{R}$.

**Lemma 1.** *There exist pairwise disjoint sets $\Sigma_\ell, \Sigma_r, \Gamma \subseteq \Sigma$, an involution $\overline{\phantom{a}} : \Gamma \to \Gamma$, and a STS $\mathcal{D} \subseteq \{(ab, \epsilon) \mid a \in \Sigma_\ell,\ b \in \Sigma_r\}$ such that $\Sigma = \Sigma_\ell \cup \Sigma_r \cup \Gamma$ and $\mathcal{R} = \mathcal{D} \cup \{(a\overline{a}, \epsilon) \mid a \in \Gamma\}$. Furthermore given $\mathcal{R}$ and $a \in \Sigma$ we can decide in uTC$^0$ whether $a$ belongs to $\Sigma_\ell$, $\Sigma_r$, or $\Gamma$.*

*Proof.* Define subsets $\Sigma_1, \Sigma_2 \subseteq \Sigma$ by $\Sigma_1 = \{a \in \Sigma \mid \exists b \in \Sigma : (ab, \epsilon) \in \mathcal{R}\}$, $\Sigma_2 = \{a \in \Sigma \mid \exists b \in \Sigma : (ba, \epsilon) \in \mathcal{R}\}$, and let $\Sigma_\ell = \Sigma_1 \backslash \Sigma_2$, $\Sigma_r = \Sigma_2 \backslash \Sigma_1$, and $\Gamma = \Sigma_1 \cap \Sigma_2$. Obviously $\Sigma_\ell$, $\Sigma_r$, and $\Gamma$ are pairwise disjoint and $\Sigma = \Sigma_\ell \cup \Sigma_r \cup \Gamma$. Now let $a \in \Gamma$. Then there exist $b, c \in \Sigma$ with $(ab, \epsilon), (ca, \epsilon) \in \mathcal{R}$. It follows $cab \to_{\mathcal{R}} b$ and $cab \to_{\mathcal{R}} c$. Since $\mathcal{R}$ is confluent we get $b = c$, i.e, $(ab, \epsilon), (ba, \epsilon) \in \mathcal{R}$ and thus $b \in \Gamma$. Now assume that also $(ab', \epsilon) \in \mathcal{R}$ for some $b' \neq b$. Then $bab' \to_{\mathcal{R}} b$ and $bab' \to_{\mathcal{R}} b'$ which contradicts the confluence of $\mathcal{R}$. Similarly there cannot exist a $b' \neq b$ with $(b'a, \epsilon) \in \mathcal{R}$. Thus we can define an involution $\overline{\phantom{a}} : \Gamma \to \Gamma$ by $\overline{a} = b$ if $(ab, \epsilon), (ba, \epsilon) \in \mathcal{R}$. The lemma follows easily. $\square$

Note that the involution $\overline{\phantom{a}} : \Gamma \to \Gamma$ may have fixed points. For the rest of this section it is helpful to eliminate these fixed points. Let $a \in \Gamma$ such that $\overline{a} = a$. Take a new symbol $a'$ and redefine the involution $\overline{\phantom{a}}$ on the alphabet $\Gamma \cup \{a'\}$ by $\overline{a} = a'$ and $\overline{a'} = a$. Let $\mathcal{R}' = (\mathcal{R} \cup \{(aa', \epsilon), (a'a, \epsilon)\}) \backslash \{(aa, \epsilon)\}$. Furthermore for $w \in \Sigma^*$ let $w' \in (\Sigma \cup \{a'\})^*$ be the word that results from $s$ by replacing the $i$th occurrence of $a$ in $w$ by $a'$ if $i$ is odd and leaving all other occurrences of symbols unchanged. Then it follows that for $s, t \in \Sigma^*$ it holds $s \overset{*}{\leftrightarrow}_{\mathcal{R}} t$ if and only if $s' \overset{*}{\leftrightarrow}_{\mathcal{R}'} t'$. Note that $s'$, $t'$, and $\mathcal{R}'$ can be calculated from $s$, $t$, and $\mathcal{R}$ in uTC$^0$. In this way we can eliminate all fixed points of the involution $\overline{\phantom{a}}$. Thus for the rest of the section we may assume that $a \neq \overline{a}$ for all $a \in \Gamma$. Let $\mathcal{S} = \{(a\overline{a}, \epsilon) \mid a \in \Gamma\} \subseteq \mathcal{R}$. Then $\Gamma^*/\mathcal{S}$ is the free group of rank $|\Gamma|/2$.

Define the morphism $\pi : \Sigma^* \to \{(,)\}^*$ by $\pi(a) = ($ for $a \in \Sigma_\ell$, $\pi(b) = )$ for $b \in \Sigma_r$, and $\pi(c) = \epsilon$ for $c \in \Gamma$. We say that a word $w \in \Sigma^*$ is *well-bracketed* if the word $\pi(w)$ is well-bracketed. It is easy to see that if $w \overset{*}{\to}_{\mathcal{R}} \epsilon$ then $w$ is well-bracketed. Furthermore Theorem 1 implies that for a word $w$ and two positions $i, j \in \{1, \ldots, |w|\}$ we can check in uTC$^0$ whether $w[i, j]$ is well-bracketed. We say that two positions $i, j \in \{1, \ldots, |w|\}$ are *corresponding brackets*, briefly $\mathrm{co}_w(i, j)$, if $i < j$, $w[i] \in \Sigma_\ell$, $w[j] \in \Sigma_r$, $w[i, j]$ is well-bracketed, and $w[i, k]$ is not well-bracketed for all $k$ with $i < k < j$. Again it can be checked in uTC$^0$, whether two positions are corresponding brackets. If $w$ is well-bracketed then we can factorize $w$ uniquely as $w = s_0 w[i_1, j_1] s_1 \cdots w[i_n, j_n] s_n$, where $n \geq 0$, $\mathrm{co}_w(i_k, j_k)$ for all $k \in \{1, \dots, n\}$ and $s_k \in \Gamma^*$ for all $k \in \{0, \dots, n\}$. We define $\mathcal{F}(w) = s_0 \cdots s_n \in \Gamma^*$.

**Lemma 2.** *The partial function $\mathcal{F} : \Sigma^* \to \Gamma^*$ (which is only defined on well-bracketed words) can be calculated in $uTC^0$.*

*Proof.* First in parallel for every $m \in \{1, \dots, |w|\}$ we calculate in $\mathrm{uTC}^0$ the value $f_m \in \{0, 1\}$, where $f_m = 0$ if and only if there exist positions $i \leq m \leq j$ such that $\mathrm{co}_w(i, j)$. Next we calculate in parallel for every $m \in \{1, \dots, |w|\}$ the sum $F_m = \sum_{i=1}^{m} f_i$, which is possible in $\mathrm{uTC}^0$ by [5]. If $F_{|w|} < m \leq |w|$ then the $m$-th output is set to the binary coding of #. If $m \leq F_{|w|}$ and $i \in \{1, \dots, |w|\}$ is such that $f_i = 1$ and $F_i = m$ then the $m$-th output is set to the binary coding of $w[i]$. ◻

**Lemma 3.** *Let $w = s_0 w[i_1, j_1] s_1 \cdots w[i_n, j_n] s_n$ be well-bracketed, where $n \geq 0$, $\mathrm{co}_w(i_k, j_k)$ for all $1 \leq k \leq n$, and $s_k \in \Gamma^*$ for all $0 \leq k \leq n$. Then $w \xrightarrow{*}_{\mathcal{R}} \epsilon$ if and only if $\mathcal{F}(w) = s_0 \cdots s_n \xrightarrow{*}_{\mathcal{S}} \epsilon$, $(w[i_k]w[j_k], \epsilon) \in \mathcal{R}$, and $w[i_k + 1, j_k - 1] \xrightarrow{*}_{\mathcal{R}} \epsilon$ for all $1 \leq k \leq n$.*

*Proof.* The if-direction of the lemma is trivial. We prove the other direction by an induction on the length of the derivation $w \xrightarrow{*}_{\mathcal{R}} \epsilon$. The case that this derivation has length 0 is trivial. Thus assume that $w = w_1 \ell w_2 \to_{\mathcal{R}} w_1 w_2 \xrightarrow{*}_{\mathcal{R}} \epsilon$. In case that the removed occurrence of $\ell$ in $w$ lies completely within one of the factors $s_k$ $(0 \leq k \leq n)$ or $w[i_k + 1, j_k - 1]$ $(1 \leq k \leq n)$ of $w$ we can directly apply the induction hypothesis to $w_1 w_2$. On the other hand if the removed occurrence of $\ell$ contains one of the positions $i_k$ or $j_k$ $(1 \leq k \leq n)$ then, since $\mathrm{co}_w(i_k, j_k)$, we must have $\ell = w[i_k]w[j_k]$, $w[i_k + 1, j_k - 1] = \epsilon$, and $w_1 w_2 = s_0 w[i_1, j_1] s_1 \cdots w[i_{k-1}, j_{k-1}](s_{k-1} s_k) w[i_{k+1}, j_{k+1}] s_{k+1} \cdots w[i_n, j_n] s_n \xrightarrow{*}_{\mathcal{R}} \epsilon$. We can conclude by using the induction hypothesis. ◻

**Lemma 4.** *For $w \in \Sigma^*$ it holds $w \xrightarrow{*}_{\mathcal{R}} \epsilon$ if and only if $w$ is well-bracketed, $\mathcal{F}(w) \xrightarrow{*}_{\mathcal{S}} \epsilon$, and for all $i, j \in \{1, \dots, |w|\}$ with $\mathrm{co}_w(i, j)$ it holds $((w[i]w[j], \epsilon) \in \mathcal{R}$ and $\mathcal{F}(w[i + 1, j - 1]) \xrightarrow{*}_{\mathcal{S}} \epsilon)$.*

*Proof.* The only if-direction can be shown by an induction on $|w|$ as follows. Let $w \xrightarrow{*}_{\mathcal{R}} \epsilon$. Then $w$ must be well-bracketed, thus we can factorize $w$ as $w = s_0 w[i_1, j_1] s_1 \cdots w[i_n, j_n] s_n$, where $n \geq 0$, $\mathrm{co}_w(i_k, j_k)$ for all $k \in \{1, \dots, n\}$, and $s_k \in \Gamma^*$ for all $k \in \{0, \dots, n\}$. By Lemma 3 above we obtain $\mathcal{F}(w) \xrightarrow{*}_{\mathcal{S}} \epsilon$, $(w[i_k]w[j_k], \epsilon) \in \mathcal{R}$, and $w[i_k + 1, j_k - 1] \xrightarrow{*}_{\mathcal{R}} \epsilon$ for all $k \in \{1, \dots, n\}$. Since $|w[i_k + 1, j_k - 1]| < |w|$ we can apply the induction hypothesis to each of the words $w[i_k + 1, j_k - 1]$ which proves the only if-direction. For the other direction assume that $w$ is well-bracketed, $\mathcal{F}(w) \xrightarrow{*}_{\mathcal{S}} \epsilon$, and for all $i, j \in \{1, \dots, |w|\}$ with $\mathrm{co}_w(i, j)$ it holds $((w[i]w[j], \epsilon) \in \mathcal{R}$ and $\mathcal{F}(w[i + 1, j - 1]) \xrightarrow{*}_{\mathcal{S}} \epsilon)$. We claim that for all $i, j \in \{1, \dots, |w|\}$ with $\mathrm{co}_w(i, j)$ it holds $w[i, j] \xrightarrow{*}_{\mathcal{R}} \epsilon$. This can be easily shown by an induction on $j - i$. Together with $\mathcal{F}(w) \xrightarrow{*}_{\mathcal{S}} \epsilon$ we get $w \xrightarrow{*}_{\mathcal{R}} \epsilon$. ◻

The previous lemma implies easily the following partial result.

**Lemma 5.** *The following problem is uTC$^0$-reducible to the WP for $F_2$.*
*INPUT: A confluent and 2-homogeneous STS $\mathcal{R}$ and a word $w \in \Sigma^*$.*
*QUESTION: Does $w \xrightarrow{*}_{\mathcal{R}} \epsilon$ (or equivalently $w \xleftrightarrow{*}_{\mathcal{R}} \epsilon$) hold?*

*Proof.* A circuit with oracle gates for the WP for $F_2$ that on input $w, \mathcal{R}$ determines whether $w \xrightarrow{*}_{\mathcal{R}} \epsilon$ can be easily built using Lemma 4. The quantification over all pairs $i, j \in \{1, \ldots, |w|\}$ in Lemma 4 corresponds to an and-gate of unbounded fan-in. In order to check whether $\mathcal{F}(w[i,j]) \xrightarrow{*}_{\mathcal{S}} \epsilon$ for two positions $i$ and $j$, we first calculate in uTC$^0$ the word $\mathcal{F}(w[i,j])$ using Lemma 2. Next we apply Corollary 1, and finally we use an oracle gate for the WP for $F_2$. $\square$

For $w \in \Sigma^*$ we define the set $\Pi(w)$ as the set of all positions $i \in \{1, \ldots, |w|\}$ such that $w[i] \in \Sigma_\ell \cup \Sigma_r$ and furthermore there does not exist a position $k > i$ with $w[i,k] \xrightarrow{*}_{\mathcal{R}} \epsilon$ and there does not exist a position $k < i$ with $w[k,i] \xrightarrow{*}_{\mathcal{R}} \epsilon$. Thus $\Pi(w)$ is the set of all positions in $w$ whose corresponding symbols are from $\Sigma_\ell \cup \Sigma_r$ but which cannot be deleted in any derivation starting from $w$. The following lemma should be compared with [20, Lemma 5.4] which makes a similar statement for arbitrary special STSs, i.e., STSs for which it is only required that each rule has the form $(s, \epsilon)$ with $s$ arbitrary.

**Lemma 6.** *For $u, v \in \Sigma^*$ let $\Pi(u) = \{i_1, \ldots, i_m\}$ and $\Pi(v) = \{j_1, \ldots, j_n\}$, where $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_n$. Define $i_0 = j_0 = 0$, $i_{m+1} = |u| + 1$, and $j_{n+1} = |v| + 1$. Then $u \xleftrightarrow{*}_{\mathcal{R}} v$ if and only if $m = n$, $u[i_k] = v[j_k]$ for $1 \le k \le n$ and $\mathcal{F}(u[i_k + 1, i_{k+1} - 1]) \xleftrightarrow{*}_{\mathcal{S}} \mathcal{F}(v[j_k + 1, j_{k+1} - 1])$ for $0 \le k \le n$.*

*Proof.* First we show the following statement:

$$\text{Let } w \in \Sigma^*. \text{ If } \Pi(w) = \emptyset \text{ then } w \text{ is well-bracketed and } w \xrightarrow{*}_{\mathcal{R}} \mathcal{F}(w). \qquad (1)$$

The case that there does not exist an $i \in \{1, \ldots, |w|\}$ with $w[i] \in \Sigma_\ell \cup \Sigma_r$ is clear. Otherwise there exists a smallest $i \in \{1, \ldots, |w|\}$ with $w[i] \in \Sigma_\ell \cup \Sigma_r$. Thus $w = s\, w[i]\, t$ for some $s \in \Gamma^*$, $t \in \Sigma^*$. Since $\Pi(w) = \emptyset$ we must have $w[i] \in \Sigma_\ell$ and there exists a minimal $j > i$ with $w[i,j] \xrightarrow{*}_{\mathcal{R}} \epsilon$. Lemma 3 implies $\mathrm{co}_w(i,j)$. Let $u$ be such that $w = s\, w[i,j]\, u$. Since $\Pi(w) = \emptyset$ we must have $\Pi(u) = \emptyset$. Inductively it follows that $u$ is well-bracketed and $u \xrightarrow{*}_{\mathcal{R}} \mathcal{F}(u)$. Thus $w$ is well-bracketed and $w \xrightarrow{*}_{\mathcal{R}} s\mathcal{F}(u) = \mathcal{F}(w)$, which proves (1).

Now we prove the lemma. Consider a factor $u_k := u[i_{k-1} + 1, i_k - 1]$ of $u$. Let $i_{k-1} < i < i_k$ such that $u[i] \in \Sigma_\ell$. Then $i \notin \Pi(u)$, hence there exists a $j > i$ such that $u[i,j] \xrightarrow{*}_{\mathcal{R}} \epsilon$. But since $i_k \in \Pi(u)$ we must have $j < i_k$. A similar argument holds if $u[i] \in \Sigma_r$, hence $\Pi(u_k) = \emptyset$ and thus $u_k \xrightarrow{*}_{\mathcal{R}} \mathcal{F}(u_k)$ by (1). We obtain $u \xrightarrow{*}_{\mathcal{R}} \mathcal{F}(u_1)u[i_1]\mathcal{F}(u_2)u[i_2] \cdots \mathcal{F}(u_m)u[i_m]\mathcal{F}(u_{m+1}) =: u'$ and similarly $v \xrightarrow{*}_{\mathcal{R}} \mathcal{F}(v_1)v[j_1]\mathcal{F}(v_2)v[j_2] \cdots \mathcal{F}(v_n)v[j_n]\mathcal{F}(v_{n+1}) =: v'$. Thus $u \xleftrightarrow{*}_{\mathcal{R}} v$ if and only if $u' \xleftrightarrow{*}_{\mathcal{R}} v'$ if and only if $u'$ and $v'$ can be reduced to a common word. But only the factors $\mathcal{F}(u_k)$ and $\mathcal{F}(v_k)$ of $u'$ and $v'$, respectively, are reducible. The lemma follows easily. $\square$

With the previous lemma the following theorem follows easily.

**Theorem 4.** *The UWP for the class of all confluent and 2-homogeneous STSs is uTC$^0$-reducible to the WP for $F_2$.*

*Proof.* Let two words $u, v \in \Sigma^*$ and a confluent and 2-homogeneous STS $\mathcal{R}$ be given. First we calculate in parallel for all $i, j \in \{1, \ldots, |u|\}$ with $i < j$ the Boolean value $e_{i,j}$, which is false if and only if $u[i,j] \xrightarrow{*}_{\mathcal{R}} \epsilon$. Next we calculate in parallel for all $i \in \{1, \ldots, |u|\}$ the number $g_i \in \{0, 1\}$ by

$$g_i = \begin{cases} 1 & \text{if } u[i] \in \Sigma_\ell \cup \Sigma_r \ \wedge \ \bigwedge_{k=1}^{i-1} e_{k,i} \ \wedge \ \bigwedge_{k=i+1}^{|u|} e_{i,k} \\ 0 & \text{else} \end{cases}$$

Thus $g_i = 1$ if and only if $i \in \Pi(u)$. Similarly we calculate for all $j \in \{1, \ldots, |v|\}$ a number $h_j \in \{0, 1\}$, which is 1 if and only if $j \in \Pi(v)$. W.l.o.g. we assume that $g_1 = g_{|u|} = h_1 = h_{|v|} = 1$, this can be enforced by appending symbols to the left and right end of $u$ and $v$. Now we calculate in parallel for all $i \in \{1, \ldots, |u|\}$ and all $j \in \{1, \ldots, |v|\}$ the sums $G_i = \sum_{k=1}^{i} g_k$ and $H_j = \sum_{k=1}^{j} h_k$, which can be done in uTC$^0$ by [5]. Finally by Lemma 6, $u \overset{*}{\leftrightarrow}_{\mathcal{R}} v$ holds if and only if $G_{|u|} = H_{|v|}$ and furthermore for all $i_1, i_2 \in \{1, \ldots, |u|\}$ and all $j_1, j_2 \in \{1, \ldots, |v|\}$ such that $(g_{i_1} = g_{i_2} = h_{j_1} = h_{j_2} = 1,\ G_{i_1} = H_{j_1}$, and $G_{i_2} = H_{j_2} = G_{i_1} + 1)$ it holds $(u[i_1] = v[j_1],\ u[i_2] = v[j_2]$, and $\mathcal{F}(u[i_1+1, i_2-1]) \overset{*}{\leftrightarrow}_{\mathcal{S}} \mathcal{F}(v[j_1+1, j_2-1]))$. Using Corollary 1, Lemma 2, and Lemma 5 the above description can be easily converted into a uTC$^0$-reduction to the WP for $F_2$. □

**Corollary 2.** *The UWP for the class of all 2-homogeneous and confluent STSs is in L. Furthermore if the WP for $F_2$ is in uNC$^1$ then the UWP for the class of all 2-homogeneous and confluent STSs is in uNC$^1$.*

## 4 The nonuniform case

In this section let $\mathcal{R}$ be a fixed 2-homogeneous STS over an alphabet $\Sigma$ which is not necessarily confluent. W.l.o.g. we may assume that $\Sigma = \{0, \ldots, n-1\}$. The following two lemmas are easy to prove.

**Lemma 7.** *Let $a, b \in \Sigma$ such that $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$ and define a morphism $\phi : \Sigma^* \to \Sigma^*$ by $\phi(a) = b$ and $\phi(c) = c$ for all $c \in \Sigma \backslash \{a\}$. Then for all $s, t \in \Sigma^*$ we have $s \overset{*}{\leftrightarrow}_{\mathcal{R}} t$ if and only if $\phi(s) \overset{*}{\leftrightarrow}_{\phi(\mathcal{R})} \phi(t)$.*

**Lemma 8.** *Let $\phi : \Sigma^* \to \Sigma^*$ be the morphism defined by $\phi(a) = \min\{b \in \Sigma \mid a \overset{*}{\leftrightarrow}_{\mathcal{R}} b\}$. Then for all $u, v \in \Sigma^*$ it holds $u \overset{*}{\leftrightarrow}_{\mathcal{R}} v$ if and only if $\phi(u) \overset{*}{\leftrightarrow}_{\phi(\mathcal{R})} \phi(v)$. Furthermore the STS $\phi(\mathcal{R})$ is confluent.*

*Proof.* All critical pairs of $\phi(\mathcal{R})$ can be resolved: If $\phi(a) \leftarrow \phi(a)\phi(b)\phi(c) \to \phi(c)$ then $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$ and thus $\phi(a) = \phi(b)$. The second statement of the lemma follows immediately from Lemma 7.

**Theorem 5.** *Let $\mathcal{R}$ be a fixed 2-homogeneous STS over an alphabet $\Sigma$. Then the WP for $\Sigma^*/\mathcal{R}$ is in L. Furthermore if the WP for $F_2$ is in uNC¹ then also the WP for $\Sigma^*/\mathcal{R}$ is in uNC¹.*

*Proof.* Let $\mathcal{R}$ be a fixed 2-homogeneous STS over an alphabet $\Sigma$ and let $\phi$ be the fixed morphism from Lemma 8. Since the morphism $\phi$ can be calculated in $\mathrm{uTC}^0$, the result follows from Corollary 2. ◻

The next theorem gives some lower bounds for word problems for 2-homogeneous STSs. It deals w.l.o.g. only with confluent and 2-homogeneous STSs. We use the notations from Lemma 1.

**Theorem 6.** *Let $\mathcal{R}$ be a confluent and 2-homogeneous STS over the alphabet $\Sigma = \Sigma_\ell \cup \Sigma_r \cup \Gamma$. Let $|\Gamma| = 2 \cdot n + f$, where $f$ is the number of fixed points of the involution $\overline{\phantom{a}} : \Gamma \to \Gamma$. If $n + f \geq 2$ but not ($n = 0$ and $f = 2$) then the WP for $\mathcal{R}$ is uNC¹-hard under DLOGTIME-reductions. If $n + f < 2$ or ($n = 0$ and $f = 2$) then the WP for $\mathcal{R}$ is in uTC⁰.*

*Proof.* If we do not remove the fixed points of the involution $\overline{\phantom{a}} : \Gamma \to \Gamma$ then the considerations from Section 3 imply that the WP for $\mathcal{R}$ is $\mathrm{uTC}^0$-reducible to the WP for $G = F_n * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$, where $*$ constructs the free product and we take $f$ copies of $\mathbb{Z}_2$ (each fixed point of $\overline{\phantom{a}}$ generates a copy of $\mathbb{Z}_2$, and the remaining $2n$ many elements in $\Gamma$ generate $F_n$). The case $n + f = 0$ is clear. If $n + f = 1$, then either $G = \mathbb{Z}$ or $G = \mathbb{Z}_2$. Both groups have a word problem in $\mathrm{uTC}^0$. If $n = 0$ and $f = 2$ then $G = \mathbb{Z}_2 * \mathbb{Z}_2$. Now $\mathbb{Z}_2 * \mathbb{Z}_2$ is a solvable group, see [23, Lemma 6.9]. Furthermore if we choose two generators $a$ and $b$ of $G$, where $a^2 = b^2 = 1$ in $G$, then the number of elements of $G$ definable by words over $\{a, b\}$ of length at most $n$ grows only polynomially in $n$, i.e, $G$ has a polynomial growth function. Now [23, Theorem 7.6] implies that the WP for $G$ is in $\mathrm{uTC}^0$. Finally let $n + f \geq 2$ but not ($n = 0$ and $f = 2$). Then $G = G_1 * G_2$, where either $G_1 \not\simeq \mathbb{Z}_2$ or $G_2 \not\simeq \mathbb{Z}_2$, hence $G$ has $F_2$ as a subgroup, see e.g. the remark in [17, p 177]. Theorem 3 implies that the WP for $G$ and thus also the WP for $\mathcal{R}$ are $\mathrm{uNC}^1$-hard under DLOGTIME-reductions. ◻

## 5 The general uniform case

In this section let $\mathcal{R}$ be an arbitrary 2-homogeneous STS over an alphabet $\Sigma$ which is not necessarily confluent. We start with some definitions. A word $w = a_1 a_2 \cdots a_n \in \Sigma^*$, where $n \geq 1$ and $a_i \in \Sigma$ for $i \in \{1, \ldots, n\}$, is an *$\mathcal{R}$-path from $a_1$ to $a_n$* if for all $i \in \{1, \ldots, n-1\}$ we have $(a_i a_{i+1}, \epsilon) \in \mathcal{R}$ or $(a_{i+1} a_i, \epsilon) \in \mathcal{R}$. Let $\triangleright$ and $\triangleleft$ be two symbols. For an $\mathcal{R}$-path $w = a_1 \cdots a_n$ the set $D_{\mathcal{R}}(w) \subseteq \{\triangleright, \triangleleft\}^*$ contains all words of the form $d_1 \cdots d_{n-1}$ such that for all $i \in \{1, \ldots, n-1\}$ if $d_i = \triangleright$ (respectively $d_i = \triangleleft$) then $(a_i a_{i+1}, \epsilon) \in \mathcal{R}$ (respectively $(a_{i+1} a_i, \epsilon) \in \mathcal{R}$). Since $\mathcal{R}$ may contain two rules of the form $(ab, \epsilon)$ and $(ba, \epsilon)$, the set $D_{\mathcal{R}}(u)$ may contain more than one word. We define a confluent and 2-homogeneous STS over $\{\triangleright, \triangleleft\}$ by $\mathcal{Z} = \{(\triangleright\triangleright, \epsilon), (\triangleleft\triangleleft, \epsilon)\}$. Finally let $[\epsilon]_{\mathcal{Z}} = \{s \in \{\triangleright, \triangleleft\}^* \mid s \xrightarrow{*}_{\mathcal{Z}} \epsilon\}$. Note that every word in $[\epsilon]_{\mathcal{Z}}$ has an even length.

**Lemma 9.** *Let $a, b \in \Sigma$. Then $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$ if and only if there exists an $\mathcal{R}$-path $w$ from $a$ to $b$ with $D_{\mathcal{R}}(w) \cap [\epsilon]_{\mathcal{Z}} \neq \emptyset$.*

*Proof.* First assume that $w = a_1 \cdots a_n$ is an $\mathcal{R}$-path such that $a_1 = a$, $a_n = b$, and $d_1 \cdots d_{n-1} \in D_{\mathcal{R}}(w) \cap [\epsilon]_{\mathcal{Z}}$. The case $n = 1$ is clear, thus assume that $n \geq 3$, $s = d_1 \cdots d_{i-1} d_{i+2} \cdots d_{n-1} \in [\epsilon]_{\mathcal{Z}}$, and $d_i = \triangleright = d_{i+1}$ (the case $d_i = \triangleleft = d_{i+1}$ is analogous). Thus $(a_i a_{i+1}, \epsilon), (a_{i+1} a_{i+2}, \epsilon) \in \mathcal{R}$ and $a_i \leftarrow a_i a_{i+1} a_{i+2} \rightarrow a_{i+2}$. Define a morphism $\varphi$ by $\varphi(a_{i+2}) = a_i$ and $\varphi(c) = c$ for all $c \in \Sigma \backslash \{a_{i+2}\}$. Then $w' = \varphi(a_1) \cdots \varphi(a_i) \varphi(a_{i+3}) \cdots \varphi(a_n)$ is a $\varphi(\mathcal{R})$-path such that $s \in D_{\varphi(\mathcal{R})}(w')$. Inductively we obtain $\varphi(a) \overset{*}{\leftrightarrow}_{\varphi(\mathcal{R})} \varphi(b)$. Finally Lemma 7 implies $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$.

Now assume that $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$ and choose a derivation $a = u_1 \leftrightarrow_{\mathcal{R}} u_2 \leftrightarrow_{\mathcal{R}} \cdots u_{n-1} \leftrightarrow_{\mathcal{R}} u_n = b$, where $n$ is minimal. The case $a = b$ is clear, thus assume that $a \neq b$ and hence $n \geq 3$. First we will apply the following transformation step to our chosen derivation: If the derivation contains a subderivation of the form $uv\ell_2 w \leftarrow u\ell_1 v\ell_2 w \rightarrow u\ell_1 vw$, where $(\ell_1, \epsilon), (\ell_2, \epsilon) \in \mathcal{R}$ then we replace this subderivation by $uv\ell_2 w \rightarrow uvw \leftarrow u\ell_1 vw$. Similarly we proceed with subderivations of the from $u\ell_2 vw \leftarrow u\ell_2 v\ell_1 w \rightarrow uv\ell_1 w$. Since the iterated application of this transformation step is a terminating process, we finally obtain a derivation $\mathcal{D}$ from $a$ to $b$ which does not allow further applications of the transformation described above. We proceed with the derivation $\mathcal{D}$. Note that $\mathcal{D}$ is also a derivation of minimal length from $a$ to $b$. Since $a$ and $b$ are both $\mathcal{R}$-irreducible, $\mathcal{D}$ must be of the form $a \overset{*}{\leftrightarrow}_{\mathcal{R}} u \leftarrow v \rightarrow w \overset{*}{\leftrightarrow}_{\mathcal{R}} b$ for some $u, v, w$. The assumptions on $\mathcal{D}$ imply that there exist $s, t \in \Sigma^*$ and $(a_1 a_2, \epsilon), (a_2 a_3, \epsilon) \in \mathcal{R}$ such that $u = sa_1 t$, $v = sa_1 a_2 a_3 t$, and $w = sa_3 t$ (or $u = sa_3 t$, $v = sa_1 a_2 a_3 t$, and $w = sa_1 t$, this case is analogous). Thus $a_1 \overset{*}{\leftrightarrow}_{\mathcal{R}} a_3$. Define the morphism $\varphi$ by $\varphi(a_3) = a_1$ and $\varphi(c) = c$ for all $c \in \Sigma \backslash \{a_3\}$. Lemma 7 implies $\varphi(a) \overset{*}{\leftrightarrow}_{\varphi(\mathcal{R})} \varphi(b)$ by a derivation which is shorter then $\mathcal{D}$. Inductively we can conclude that there exists a $\varphi(\mathcal{R})$-path $w'$ from $\varphi(a)$ to $\varphi(b)$ with $D_{\varphi(\mathcal{R})}(w') \cap [\epsilon]_{\mathcal{Z}} \neq \emptyset$. By replacing in the path $w'$ some occurrences of $a_1$ by one of the $\mathcal{R}$-paths $a_1 a_2 a_3$, $a_3 a_2 a_1$, or $a_3 a_2 a_1 a_2 a_3$, we obtain an $\mathcal{R}$-path $w$ from $a$ to $b$. For instance if $w'$ contains a subpath of the form $ca_1 d$, where $(ca_1, \epsilon), (a_1 d, \epsilon) \notin \mathcal{R}$ but $(ca_3, \epsilon), (a_3 d, \epsilon) \in \mathcal{R}$, then we replace $ca_1 d$ by $ca_3 a_2 a_1 a_2 a_3 d$. Since for all $v \in \{a_1 a_2 a_3, a_3 a_2 a_1, a_3 a_2 a_1 a_2 a_3\}$ we have $D_{\mathcal{R}}(v) \cap [\epsilon]_{\mathcal{Z}} \neq \emptyset$ it follows $D_{\mathcal{R}}(w) \cap [\epsilon]_{\mathcal{Z}} \neq \emptyset$. $\square$

Define the set $\mathcal{I}$ by $\mathcal{I} = \{s \in [\epsilon]_{\mathcal{Z}} \backslash \{\epsilon\} \mid \forall p, q \in \Sigma^* \backslash \{\epsilon\} : s = pq \Rightarrow p \notin [\epsilon]_{\mathcal{Z}}\}$. The following lemma follows immediately from the definition of $\mathcal{I}$.

**Lemma 10.** *It holds $\mathcal{I} \subseteq \triangleright\{\triangleright, \triangleleft\}^* \triangleright \cup \triangleleft \{\triangleright, \triangleleft\}^* \triangleleft$ and $[\epsilon]_{\mathcal{Z}} = \mathcal{I}^*$.*

Define a binary relation $T \subseteq \Sigma \times \Sigma$ by $(a, b) \in T$ if and only if there exists an $\mathcal{R}$-path $w$ from $a$ to $b$ with $|w|$ odd and furthermore there exist $c, d \in \Sigma$ such that either $(ac, \epsilon), (db, \epsilon) \in \mathcal{R}$ or $(ca, \epsilon), (bd, \epsilon) \in \mathcal{R}$. Note that $T$ is symmetric.

**Lemma 11.** *Let $a, b \in \Sigma$. Then $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$ if and only if $(a, b) \in \overset{*}{T}$.*

*Proof.* For the if-direction it suffices to show that $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$ if $(a, b) \in T$. Thus assume that there exists an $\mathcal{R}$-path $w$ from $a$ to $b$ with $|w|$ odd and furthermore there exist $c, d \in \Sigma$ such that $(ac, \epsilon), (db, \epsilon) \in \mathcal{R}$ (the case that $(ca, \epsilon), (bd, \epsilon) \in \mathcal{R}$ is analogous). Let $s \in D_{\mathcal{R}}(w)$. Since $(ac, \epsilon), (db, \epsilon) \in \mathcal{R}$, also the word $w_i = (ac)^{|s|} w (db)^i$ is an $\mathcal{R}$-path for every $i \geq 0$. It holds $s_i = (\triangleright\triangleleft)^{|s|} s (\triangleleft\triangleright)^i \in D_{\mathcal{R}}(w_i)$. Since $|s|$ is even and $|s| < |(\triangleright\triangleleft)^{|s|}|$, the (unique) $\mathcal{Z}$-normalform of the prefix $(\triangleright\triangleleft)^{|s|} s$ of $s_i$ has the form $(\triangleright\triangleleft)^j$ for some $j \geq 0$. Thus $s_j \in [\epsilon]_{\mathcal{Z}}$ and $D_{\mathcal{R}}(w_j) \cap [\epsilon]_{\mathcal{Z}} \neq \emptyset$. By Lemma 9 we have $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$.

Now let $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$. By Lemma 9 there exists an $\mathcal{R}$-path $w$ from $a$ to $b$ and a word $s \in D_{\mathcal{R}}(w) \cap [\epsilon]_{\mathcal{Z}}$. Let $s = s_1 \cdots s_m$ where $m \geq 0$ and $s_i \in \mathcal{I}$. Let $w_i$ be a subpath of $w$ which goes from $a_i$ to $a_{i+1}$ such that $s_i \in D_{\mathcal{R}}(w_i)$ and $a_1 = a$, $a_{m+1} = b$. It suffices to show that $(a_i, a_{i+1}) \in T$. Since $s_i \in \mathcal{I} \subseteq [\epsilon]_{\mathcal{Z}}$, the length of $s_i$ is even. Thus $|w_i|$ is odd. Next $s_i \in \triangleright\{\triangleright, \triangleleft\}^* \triangleright \cup \triangleleft \{\triangleright, \triangleleft\}^* \triangleleft$ by Lemma 10. Let $s_i \in \triangleright\{\triangleright, \triangleleft\}^*\triangleright$, the other case is symmetric. Hence there exist rules $(a_i c, \epsilon), (d a_{i+1}, \epsilon) \in \mathcal{R}$. Thus $(a_i, a_{i+1}) \in T$. □

The preceding lemma is the key for proving that the UWP for the class of 2-homogeneous STSs is in SL. In general it is quite difficult to prove that a problem is contained in SL. A useful strategy developed in [12] and applied in [25, 26] is based on a logical characterization of SL. In the following we consider finite structures of the form $\mathcal{A} = (\{0, \ldots, n-1\}, 0, \max, s, R)$. Here $\max = n - 1$, and $R$ and $s$ are binary relations on $\{0, \ldots, n-1\}$, where $s(x, y)$ holds if and only if $y = x + 1$. The logic FO+posSTS is the set of all formulas build up from the constant 0 and max, first-order variables $x_1, x_2, \ldots$, the binary relations $s$ and $R$, the equality $=$, the Boolean connectives $\neg, \wedge$, and $\vee$, the quantifiers $\forall$ and $\exists$, and the symmetric transitive closure operator STC, where STC is not allowed to occur within a negation $\neg$. The semantic of STC is the following. Let $\varphi(x, y)$ be a formula of FO+posSTS with two free variables $x$ and $y$, and let $\mathcal{A} = (\{0, \ldots, n-1\}, 0, \max, s, R)$ be a structure. Assume that $\varphi(x, y)$ describes the binary relation $S$ over $\{0, \ldots, n-1\}$, i.e, $\mathcal{A} \models \varphi(i, j)$ if and only if $(i, j) \in S$ for all $i, j \in \{0, \ldots, n-1\}$. Then $[\mathrm{STC}x, y\ \varphi(x, y)]$ is a formula of FO+posSTS with two free variables, and for all $i, j \in \{0, \ldots, n-1\}$ it holds $\mathcal{A} \models [\mathrm{STC}x, y\ \varphi(x, y)](i, j)$ if and only if $(i, j)$ belongs to the symmetric, transitive, and reflexive closure of $S$, i.e, $(i, j) \in (S \cup S^{-1})^*$. In [12] it was shown that for every fixed variable-free formula $\varphi$ of FO+posSTC the following problem belongs to SL:

INPUT: A binary coded structure $\mathcal{A} = (\{0, \ldots, n-1\}, 0, \max, s, R)$
QUESTION: Does $\mathcal{A} \models \varphi$ hold?

**Theorem 7.** *The following problem is SL-complete:*
*INPUT: A 2-homogeneous STS $\mathcal{R}$ over an alphabet $\Sigma$ and $a, b \in \Sigma$.*
*QUESTION: Does $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$ hold?*

*Proof.* First we show containment in SL. Let $\mathcal{R}$ be a 2-homogeneous STS over an alphabet $\Sigma$ and let $a, b \in \Sigma$. W.l.o.g. we may assume that $\Sigma = \{0, \ldots, n-1\}$ and $a = 0, b = n-1$. If $a = 0$ and $b = n-1$ does not hold then it can be enforced by relabeling the alphabet symbols. This relabeling can be done in deterministic log-space and we can use the fact that $\mathrm{L}^{\mathrm{SL}} = \mathrm{SL}$. We identify the input $\mathcal{R}$, $a$, $b$ with the structure $\mathcal{A} = (\Sigma, 0, \max, s, R)$, where $R = \{(i, j) \mid (ij, \epsilon) \in \mathcal{R}\}$. Now define formulas $S(x,y)$ and $T(x,y)$ as follows:

$$S(x,y) \quad :\Leftrightarrow \quad \exists z\{(R(x,z) \ \vee \ R(z,x)) \ \wedge \ (R(y,z) \ \vee \ R(z,y))\}$$

$$T(x,y) \quad :\Leftrightarrow \quad [STCu,v\ S(u,v)](x,y) \ \wedge \ \exists x',y' \left\{ \begin{array}{l} (R(x,x') \ \wedge \ R(y',y)) \vee \\ (R(x',x) \ \wedge \ R(y,y')) \end{array} \right\}$$

By Lemma 11, $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$ if and only if $\mathcal{A} \models [STCu,v\ T(u,v)](0,\max)$. Thus containment in SL follows from [12]. In order to show SL-hardness we use the SL-complete undirected graph accessibility problem (UGAP), see also [14]:

INPUT: An undirected graph $G=(V,E)$ and two nodes $a,b \in V$.

QUESTION: Does there exist a path in $G$ from $a$ to $b$?

Let $G=(V,E),a,b$ be an instance of UGAP, where $E \subseteq \{\{v,w\} \mid v,w \in V\}$ and of course $V \cap E = \emptyset$. We define a 2-homogeneous STS $\mathcal{R}$ over $V \cup E$ by $\mathcal{R} = \{(ce,\epsilon), (ec,\epsilon) \mid c \in V, e \in E, c \in e\}$. We claim that there exists a path in $G$ from $a$ to $b$ if and only if $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$. First assume that there exists a path $a=a_1,a_2,\cdots,a_n=b$ with $\{a_i,a_{i+1}\}=e_i \in E$. The case $n=1$ is clear. If $n>1$ then by induction $a \overset{*}{\leftrightarrow}_{\mathcal{R}} a_{n-1}$. Thus $a \overset{*}{\leftrightarrow}_{\mathcal{R}} a_{n-1} \leftarrow a_{n-1}e_{n-1}a_n \rightarrow a_n = b$. Conversely assume that $a$ and $b$ belong to different connected components of $G$. Let $V_a$ and $E_a$ be the set of all nodes and edges, respectively, that belong to the connected component of $a$. Define a projection $\pi : V \cup E \to V_a \cup E_a$ by $\pi(x)=\epsilon$ if $x \notin V_a \cup E_a$ and $\pi(x)=x$ if $x \in V_a \cup E_a$. If $a \overset{*}{\leftrightarrow}_{\mathcal{R}} b$ then $a=\pi(a) \overset{*}{\leftrightarrow}_{\pi(\mathcal{R})} \pi(b)=\epsilon$, which is impossible since $u \overset{*}{\leftrightarrow}_{\pi(\mathcal{R})} \epsilon$ implies $|u|_V = |u|_E$, where $|u|_X$ is the number of occurrences of symbols from $X$ in $u$. $\square$

**Theorem 8.** *The UWP for the class of all 2-homogeneous STSs is SL-complete.*

*Proof.* By Theorem 7 it remains to show containment in SL. W.l.o.g. let $\Sigma = \{0,\ldots,n-1\}$. Let $\phi$ be the morphism from Lemma 8. We check whether $u \overset{*}{\leftrightarrow}_{\mathcal{R}} v$ by essentially running the log-space algorithm for the UWP for confluent and 2-homogeneous STSs from Section 3, but each time we read from the input-tape (the binary coding of) a symbol $a \in \Sigma$, we replace $a$ by $\phi(a)$. Since $\phi(a) = \min\{b \in \Sigma \mid a \overset{*}{\leftrightarrow}_{\mathcal{R}} b\}$, Theorem 7 implies that we can find $\phi(a)$ by at most $n$ queries to an SL-oracle. Since $\mathrm{L}^{\mathrm{SL}} = \mathrm{SL}$, the theorem follows. $\square$

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