Automatic Structures of Bounded Degree

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Abstract. The first-order theory of an automatic structure is known to be decidable but there are examples of automatic structures with nonelementary first-order theories. We prove that the first-order theory of an automatic structure of bounded degree (meaning that the corresponding Gaifman-graph has bounded degree) is elementary decidable. More precisely, we prove an upper bound of triply exponential alternating time with a linear number of alternations. We also present an automatic structure of bounded degree such that the corresponding first-order theory has a lower bound of doubly exponential time with a linear number of alternations. We prove similar results also for tree automatic structures.

1 Introduction

Automatic structures were introduced in [13, 15]. The idea goes back to the concept of automatic groups [8]. Roughly speaking, a structure is called automatic if the elements of the universe can be represented as words from a regular language and every relation of the structure can be recognized by a finite state automaton with several heads that proceed synchronously. Automatic structures received increasing interest during the last years [1, 3, 14, 16–18]. One of the main motivations for investigating automatic structures is the fact that every automatic structure has a decidable first-order theory. On the other hand it is known that there exist automatic structures with a nonelementary first-order theory [3]. This motivates the search for subclasses of automatic structures for which the first-order theory becomes elementary decidable. In this paper we will present such a subclass, namely automatic structures of bounded degree, where the bounded degree property refers to the Gaifman-graph of the structure. Using a method of Ferrante and Rackoff [9] (see Section 3), we show in Section 4 that for every automatic structure of bounded degree the first-order theory can be decided in triply exponential alternating time with a linear number of alternations (Theorem 3). We are currently not able to match this upper bound by a sharp lower bound. But in Section 6 we will construct an example of an automatic structure of bounded degree such that the corresponding first-order theory has a lower bound of doubly exponential time with a linear number of alternations (Theorem 5). Finally, in Section 7 we will briefly discuss the extension of our results from Section 4 to tree automatic structures [2].

2 Preliminaries

General notations Let Γ be a finite alphabet and $w \in \Gamma^*$ be a finite word over Γ. The length of w is denoted by $|w|$. We also write $\Gamma^n = \{w \in \Gamma^* \mid n = |w|\}$ and $\Gamma^{\leq n} = \{w \in \Gamma^* \mid n \geq |w|\}$. Let us define $\exp(0,x) = x$ and $\exp(n+1,x) = 2^{\exp(n,x)}$ for $x \in \mathbb{N}$. A computational problem is called *elementary* if it can be solved in time $\exp(c,n)$ for some constant $c \in \mathbb{N}$.

In this paper we will deal with alternating complexity classes, see [5, 19] for more details. Roughly speaking, an *alternating Turing-machine* is a nondeterministic Turing-machine, where the set of states is partitioned into existential and universal states. A configuration with a universal (resp. existential) state is accepting if every (resp. some) successor state is accepting. An alternation in a computation of an alternating Turing-machine is a transition from a universal state to an existential state or vice versa. For functions $t(n)$ and $a(n)$ with $a(n) \leq t(n)$ for all $n \geq 0$ let $\mathrm{ATIME}(a(n), t(n))$ denote the class of all problems that can be solved on an alternating Turing-machine in time $t(n)$ with at most $a(n)$ alternations. It is known that $\mathrm{ATIME}(t(n), t(n))$ is contained in $\mathrm{DSPACE}(t(n))$ if $t(n) \geq n$ [5].

Structures The notion of a structure (or model) is defined as usual in logic. Here we only consider *relational structures*. Sometimes, we will also use constants, but a constant c can be always replaced by the unary relation $\{c\}$. Let us fix a relational structure $\mathcal{A} = (A, (R_i)_{i\in J})$, where $R_i \subseteq A^{n_i}$, $i \in J$. For $B \subseteq A$ we define the restriction $\mathcal{A}\restriction B = (B, (R_i \cap B^{n_i})_{i\in J})$. Given further constants $c_1, \ldots, c_n \in A$, we write $(\mathcal{A}, c_1, \ldots, c_n)$ for the structure $(A, (R_i)_{i\in J}, c_1, \ldots, c_n)$.

The *Gaifman-graph* $G_\mathcal{A}$ of the structure $\mathcal{A}$ is the following undirected graph:

$$G_\mathcal{A} = (A, \{(a,b) \in A \times A \mid \bigvee_{i\in J} \exists (c_1, \ldots, c_{n_i}) \in R_i \; \exists j,k : c_j = a \neq b = c_k\}).$$

Thus, the set of nodes is the universe of $\mathcal{A}$ and there is an edge between two elements, if and only if they are contained in some tuple belonging to one of the relations of $\mathcal{A}$. With $d_\mathcal{A}(a,b)$, where $a, b \in A$, we denote the distance between a and b in $G_\mathcal{A}$, i.e., it is the length of a shortest path connecting a and b in $G_\mathcal{A}$. For $a \in A$ and $r \geq 0$ we denote with $S_\mathcal{A}(r,a) = \{b \in A \mid d_\mathcal{A}(a,b) \leq r\}$ the r-sphere around a. If $\widetilde{a} = (a_1, \ldots, a_n) \in A^n$ is a tuple, then $S_\mathcal{A}(r, \widetilde{a}) = \bigcup_{i=1}^n S_\mathcal{A}(r, a_i)$. The substructure of $\mathcal{A}$ that is induced by $S_\mathcal{A}(r, \widetilde{a})$ is denoted by $N_\mathcal{A}(r, \widetilde{a})$, i.e., $N_\mathcal{A}(r, \widetilde{a}) = \mathcal{A}\restriction S_\mathcal{A}(r, \widetilde{a})$. A *connected component* of the structure $\mathcal{A}$ is any induced substructure $\mathcal{A}\restriction C$, where C is a connected component of the Gaifman-graph $G_\mathcal{A}$. We say that the structure $\mathcal{A}$ has *bounded degree*, if its Gaifman-graph $G_\mathcal{A}$ has bounded degree, i.e., there exists a constant d such that every $a \in A$ is adjacent to at most d other nodes in $G_\mathcal{A}$.

First-order logic For more details concerning first-order logic see e.g. [12]. Let us fix a structure $\mathcal{A} = (A, (R_i)_{i\in J})$, where $R_i \subseteq A^{n_i}$. The *signature of* $\mathcal{A}$ contains for each $i \in J$ a relation symbol of arity n_i that we denote without risk of confusion by R_i as well. Let $\mathbb{V}$ be a countable infinite set of variables, which evaluate to elements from the universe A. *First-order formulas* over the signature of $\mathcal{A}$ are constructed from the atomic formulas $x = y$ and $R_i(x_1, \ldots, x_{n_i})$, where $i \in J$ and $x, y, x_1, \ldots, x_{n_i} \in \mathbb{V}$, using Boolean connectives and quantifications over variables from $\mathbb{V}$. The notion of a free variable is defined as usual. The *quantifier-depth* of a formula ϕ is the maximal number of nested quantifiers in ϕ. A first-order formula without free variables is called a *first-order sentence*. If $\varphi(x_1, \ldots, x_n)$ is a first-order formula with free variables among $x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in A$, then $\mathcal{A} \models \varphi(a_1, \ldots, a_n)$ means that φ evaluates to true in $\mathcal{A}$ when the free variable x_i evaluates to a_i. The *first-order theory* of $\mathcal{A}$, denoted by $\mathrm{FOTh}(\mathcal{A})$, is the set of all first-order sentences φ such that $\mathcal{A} \models \varphi$. Given a formula $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $b_1, \ldots, b_m \in A$, $\varphi(x_1, \ldots, x_n, b_1, \ldots, b_m)^{\mathcal{A}}$ denotes the n-ary relation $\{(a_1, \ldots, a_n) \in A^n \mid \mathcal{A} \models \varphi(a_1, \ldots, a_n, b_1, \ldots, b_m)\}$. Let Σ be an arbitrary set of first-order sentences over some fixed signature $\mathcal{S}$. A *model* of Σ is a structure $\mathcal{A}$ with signature $\mathcal{S}$ such that $\mathcal{A} \models \psi$ for every $\psi \in \Sigma$. With $\mathrm{sat}(\Sigma)$ we denote the set of all first-order sentences ϕ over the signature $\mathcal{S}$ such that $\mathcal{A} \models \phi$ for some model $\mathcal{A}$ of Σ. The set of all sentences ϕ such that $\mathcal{A} \models \phi$ for every model $\mathcal{A}$ of Σ is denoted by $\mathrm{val}(\Sigma)$. Note that if Σ is *complete*, i.e., for every first-order sentence ϕ either $\phi \in \Sigma$ or $\neg\phi \in \Sigma$ (this holds in particular if $\Sigma = \mathrm{FOTh}(\mathcal{A})$ for some structure $\mathcal{A}$), then $\mathrm{sat}(\Sigma) = \mathrm{val}(\Sigma)$.

Let C be some complexity class. We say that C is a *hereditary lower bound* for a theory $\mathrm{FOTh}(\mathcal{A})$ if for every $\Sigma \subseteq \mathrm{FOTh}(\mathcal{A})$ neither $\mathrm{sat}(\Sigma)$ nor $\mathrm{val}(\Sigma)$ is in C [6]. Thus, in particular $\mathrm{FOTh}(\mathcal{A})$ does not belong to the class C.

Automatic structures See [3, 15] for more details concerning automatic structures. Let us fix $n \in \mathbb{N}$ and a finite alphabet Γ. Let $\# \notin \Gamma$ be an additional padding symbol. For words $w_1, \ldots, w_n \in \Gamma^*$ we define the *convolution* $w_1 \otimes w_2 \otimes \cdots \otimes w_n$, which is a word over the alphabet $\prod_{i=1}^{n}(\Gamma \cup \{\#\})$, as follows: Let $w_i = a_{i,1}a_{i,2} \cdots a_{i,k_i}$ with $a_{i,j} \in \Gamma$ and $k = \max\{k_1, \ldots, k_n\}$. For $k_i < j \leq k$ define $a_{i,j} = \#$. Then

$$w_1 \otimes \cdots \otimes w_n = (a_{1,1}, \ldots, a_{n,1}) \cdots (a_{1,k}, \ldots, a_{n,k}).$$

Thus, for instance $aba \otimes bbabb = (a,b)(b,b)(a,a)(\#,b)(\#,b)$. An n-ary relation $R \subseteq (\Gamma^*)^n$ is called automatic if the language $\{w_1 \otimes \cdots \otimes w_n \mid (w_1, \ldots, w_n) \in R\}$ is a regular language.

Now let $\mathcal{A} = (A, (R_i)_{i \in J})$ be an arbitrary relational structure with finitely many relations, where $R_i \subseteq A^{n_i}$. A tuple (Γ, L, h) is called an *automatic presentation* for $\mathcal{A}$ if

- Γ is a finite alphabet,
- $L \subseteq \Gamma^*$ is a regular language,
- $h : L \to A$ is a surjective function,
- the relation $\{(u, v) \in L \times L \mid h(u) = h(v)\}$ is automatic, and
- the relation $\{(u_1, \ldots, u_{n_i}) \in L^{n_i} \mid (h(u_1), \ldots, h(u_{n_i})) \in R_i\}$ is automatic for every $i \in J$.

We say that $\mathcal{A}$ is *automatic* if there exists an automatic presentation for $\mathcal{A}$. The following result from [15] can be shown by induction on the structure of the formula φ.

Proposition 1 (cf [15]). *Let (Γ, L, h) be an automatic presentation for the structure $\mathcal{A}$ and let $\varphi(x_1, \dots, x_n)$ be a first-order formula over the signature of $\mathcal{A}$. Then the relation $\{(u_1, \dots, u_n) \in L^n \mid \mathcal{A} \models \varphi(h(u_1), \dots, h(u_n))\}$ is automatic.*

This proposition implies the following result, which is one of the main motivations for investigating automatic structures.

Theorem 1 (cf [15]). *If $\mathcal{A}$ is automatic, then* $\mathrm{FOTh}(\mathcal{A})$ *is decidable.*

In [3] it was shown that even the extension of first-order logic, which allows to say that there are infinitely many x with $\phi(x)$, is decidable. On the other hand there are automatic structures with a nonelementary first-order theory [3]. For instance the structure $(\{0,1\}^*, s_0, s_1, \preceq)$, where $s_i(w) = wi$ for $w \in \{0,1\}^*$ and $i \in \{0,1\}$ and $\preceq$ is the prefix order on finite words, has a nonelementary first-order theory, see e.g. [6, Example 8.3]. In Section 4 we will show that for automatic structures of bounded degree this cannot happen: in this case the first-order theory is in $\mathrm{ATIME}(O(n), \exp(3, O(n)))$.

Let us end this section with two typical examples for automatic structures of bounded degree:

Transition graphs of machines like Turing-machines or counter machines: Let $\mathcal{M}$ be such a machine, $\mathcal{C}(\mathcal{M})$ the set of all possible configurations of $\mathcal{M}$, and $\Rightarrow_{\mathcal{M}}$ the one-step transition relation between configurations. Then $(\mathcal{C}(\mathcal{M}), \Rightarrow_{\mathcal{M}})$ is the transition graph of $\mathcal{M}$ and easily seen to be automatic.

Cayley-graphs of automatic groups [8] or more general *Cayley-graphs of automatic monoids of finite geometric type* [20]: Let $\mathcal{M} = (M, \circ)$ be a finitely generated monoid and Γ a finite generating set for $\mathcal{M}$. Then the Cayley-graph of $\mathcal{M}$ with respect to Γ is the structure $(M, (\{(x, x \circ a) \mid x \in M, a \in \Gamma\})_{a \in \Gamma})$. It can be viewed as a Γ-labeled directed graph: there is an a-labeled edge from x to y if and only if $y = x \circ a$. Automatic monoids [4] have the property that their Cayley-graphs are automatic, but in general these graphs may have unbounded degree (more precisely, a node may have unbounded indegree). On the other hand, if the Cayley-graph of $\mathcal{M}$ has bounded degree with respect to some finite generating set, then it is easy to see that this holds for every finite generating set of $\mathcal{M}$. In this case, the monoid $\mathcal{M}$ is of finite geometric type [20]. This is in particular the case for right-cancellative monoids and hence for groups.

Moreover, the class of automatic structures of bounded degree is closed under operations like for instance disjoint union or direct product [3].

3 The method of Ferrante and Rackoff

In order to prove that the first-order theory of an automatic structure of bounded degree is elementary, we have to introduce a general method from [9].

Let us fix a structure $\mathcal{A}$ with universe A. Roughly speaking, Gaifman's Theorem [11] states that first-order logic only allows to express local properties of structures, see [7] for a recent account of this result. For our use, the following weaker statement is sufficient, which is an immediate consequence of the main theorem in [11].

Theorem 2 (cf. [11]). *Let $\widetilde{a} = (a_1, a_2, \ldots, a_k)$ and $\widetilde{b} = (b_1, b_2, \ldots, b_k)$, where $a_i, b_i \in A$, such that $(N_{\mathcal{A}}(7^n, \widetilde{a}), \widetilde{a}) \cong (N_{\mathcal{A}}(7^n, \widetilde{b}), \widetilde{b})$.*[1] *Then, for any first-order formula $\varphi(x_1, \ldots, x_k)$ of quantifier-depth at most n, we have $\mathcal{A} \models \varphi(\widetilde{a})$ if and only if $\mathcal{A} \models \varphi(\widetilde{b})$.*

A *norm function* on $\mathcal{A}$ is just a function $\lambda : A \to \mathbb{N}$. Let us fix a norm function λ on $\mathcal{A}$. We write $\mathcal{A} \models \exists x \leq n : \varphi$ in order to express that there exists $a \in A$ such that $\lambda(a) \leq n$ and $\mathcal{A} \models \varphi(a)$, and similarly for $\forall x \leq n : \varphi$. Let $H : \{(j, d) \in \mathbb{N} \times \mathbb{N} \mid j \leq d\} \to \mathbb{N}$ be a function such that the following holds: For all $j \leq d \in \mathbb{N}$, all $\widetilde{a} = (a_1, a_2, \ldots, a_{j-1}) \in A^{j-1}$ with $\lambda(a_i) \leq H(i, d)$, and all $a \in A$, there exists $a_j \in A$ with $\lambda(a_j) \leq H(j, d)$ and

$$(N_{\mathcal{A}}(7^{d-j}, \widetilde{a}, a), \widetilde{a}, a) \cong (N_{\mathcal{A}}(7^{d-j}, \widetilde{a}, a_j), \widetilde{a}, a_j).$$

Then $\mathcal{A}$ is called *H-bounded* (with respect to the norm function λ). This definition is a slight variant of the definition in [9] that suits our needs much better than the original formulation. The following corollary to Theorem 2 was shown by Ferrante and Rackoff for their version of H-bounded structures.

Corollary 1 (cf. [9]). *Let $\mathcal{A}$ be a relational structure with universe A and norm λ and let $H : \{(j, d) \in \mathbb{N} \times \mathbb{N} \mid j \leq d\} \to \mathbb{N}$ be a function such that $\mathcal{A}$ is H-bounded. Then for any first-order formula $\varphi \equiv Q_1 x_1\, Q_2 x_2 \cdots Q_d x_d : \psi$ where ψ is quantifier free and $Q_i \in \{\exists, \forall\}$, we have $\mathcal{A} \models \varphi$ if and only if*

$$\mathcal{A} \models Q_1 x_1 \leq H(1, d)\, Q_2 x_2 \leq H(2, d) \cdots Q_d x_d \leq H(d, d) : \psi.$$

Proof. For $j \leq d$, let ψ_j denote the formula $Q_j x_j\, Q_{j+1} x_{j+1} \cdots Q_d x_d : \psi$ and let φ_j stand for the sentence

$$Q_1 x_1 \leq H(1, d) \cdots Q_{j-1} x_{j-1} \leq H(j-1, d)\, \psi_j.$$

Thus, $\varphi_1 \equiv \varphi$. We show that $\mathcal{A} \models \varphi_j$ if and only if $\mathcal{A} \models \varphi_{j+1}$, which then proves the corollary.

Let $\widetilde{a} = (a_1, \ldots, a_{j-1}) \in A^{j-1}$ with $\lambda(a_i) \leq H(i, d)$. First assume $Q_j = \exists$, i.e., $\psi_j \equiv \exists x_j : \psi_{j+1}$. If $\mathcal{A} \models \psi_j(\widetilde{a})$, then there is $a \in A$ with $\mathcal{A} \models \psi_{j+1}(\widetilde{a}, a)$. By our assumption on the norm function λ, we find $a_j \in A$ with $\lambda(a_j) \leq H(j, d)$ and

$$(N_{\mathcal{A}}(7^{d-j}, \widetilde{a}, a), \widetilde{a}, a) \cong (N_{\mathcal{A}}(7^{d-j}, \widetilde{a}, a_j), \widetilde{a}, a_j). \tag{1}$$

[1] Thus, there exists a bijection $f : S_{\mathcal{A}}(7^n, \widetilde{a}) \to S_{\mathcal{A}}(7^n, \widetilde{b})$, which preserves all relations from $\mathcal{A}$ and such that $f(a_i) = b_i$ for $1 \leq i \leq k$.

Since the quantifier depth of ψ_{j+1} is $d-j$, Theorem 2 implies $\mathcal{A} \models \psi_{j+1}(\widetilde{a}, a_j)$. Thus, $\mathcal{A} \models (\exists x_j \leq H(j,d) : \psi_{j+1})(\widetilde{a})$. If, conversely, $\mathcal{A} \models (\exists x_j \leq H(j,d) : \psi_{j+1})(\widetilde{a})$, we have trivially $\mathcal{A} \models \psi_j(\widetilde{a})$.

Assume now that $Q_j = \forall$, i.e., $\psi_j \equiv \forall x_j : \psi_{j+1}$. If $\mathcal{A} \models \psi_j(\widetilde{a})$, then of course also $\mathcal{A} \models (\forall x_j \leq H(j,d) : \psi_{j+1})(\widetilde{a})$. Now assume that

$$\mathcal{A} \models (\forall x_j \leq H(j,d) : \psi_{j+1})(\widetilde{a}) \tag{2}$$

and let $a \in A$ be arbitrary. We have to show that $\mathcal{A} \models \psi_{j+1}(\widetilde{a}, a)$. The case $\lambda(a) \leq H(j,d)$ is clear. Thus, assume that $\lambda(a) > H(j,d)$. Then there exists $a_j \in A$ with $\lambda(a_j) \leq H(j,d)$ and (1). Since $\lambda(a_j) \leq H(j,d)$, (2) implies $\mathcal{A} \models \psi_{j+1}(\widetilde{a}, a_j)$. Finally, Theorem 2 implies $\mathcal{A} \models \psi_{j+1}(\widetilde{a}, a)$. □

4 An upper bound

In this section we apply the method of Ferrante and Rackoff in order to prove the following result:

Theorem 3. *If $\mathcal{A}$ is an automatic structure of bounded degree, then* $\mathrm{FOTh}(\mathcal{A})$ *can be decided in* $\mathrm{ATIME}(O(n), \exp(3, O(n)))$.

Proof. Fix an automatic presentation (Γ, L, h) for $\mathcal{A}$ and let the degree of the Gaifman-graph $G_{\mathcal{A}}$ be bounded by δ. By [15] we can assume that $h : L \to \mathcal{A}$ is injective and thus bijective. Hence, we may assume that L is the universe of $\mathcal{A}$ (and h is the identity function). Let E be the edge relation of $G_{\mathcal{A}}$. Since this relation is first-order definable in $\mathcal{A}$, Proposition 1 implies that the relation E is automatic. Let γ be the number of states of a finite automaton A_E for the language $\{u \otimes v \mid (u,v) \in E\}$.

Claim 1. If $(u,v) \in E$, then $|(|u| - |v|)| \leq \gamma$.

In order to deduce a contradiction, assume w.l.o.g. that $(u,v) \in E$ and $|v| - |u| > \gamma$. Then a simple pumping argument shows that the automaton A_E accepts an infinite number of words of the form $u \otimes w$ with $w \in L$ and $|w| \geq |v|$. It follows that the Gaifman-graph $G_{\mathcal{A}}$ has infinite degree, which is a contradiction.

Claim 2. Let $r \in \mathbb{N}$ and $u \in L$. Then there exists a finite automaton $A_{r,u}$ with $\exp(2, O(r))$ many states such that

$$L(A_{r,u}) = \{v \in L \mid (N_{\mathcal{A}}(r,u), u) \cong (N_{\mathcal{A}}(r,v), v)\}.$$

Thus, the automaton $A_{r,u}$ accepts a word $v \in L$ if and only if the r-sphere around v is isomorphic to the r-sphere around u (with u mapped to v). For the proof of Claim 2 first notice that since $G_{\mathcal{A}}$ has bounded degree, $|S_{\mathcal{A}}(r,u)| \in 2^{O(r)}$. We will use this in order to describe the finite substructure $N_{\mathcal{A}}(r,u)$ by a formula of size $2^{O(r)}$ over the signature of $\mathcal{A}$:

First, for $0 \leq n \leq \delta$ (δ bounds the degree of the Gaifman-graph) let the formula $\deg_n(x)$ express that the degree of x in the Gaifman-graph $G_{\mathcal{A}}$ is exactly

n. Thus, $\deg_n(x)$ is a fixed first-order formula over the signature of $\mathcal{A}$. Next take $m = |S_{\mathcal{A}}(r,u)| \in 2^{O(r)}$ many variables $x_1, \ldots, x_m$, where x_i represents the element $u_i \in S_{\mathcal{A}}(r,u)$ ($u_i \neq u_j$ for $i \neq j$) and w.l.o.g. $u = u_1$. Then write down the conjunction of the following formulas, where R is an arbitrary relation of $\mathcal{A}$ and $0 \leq n \leq \delta$:

- $x_i \neq x_j$ for $i \neq j$,
- $R(x_{i_1}, \ldots, x_{i_n})$ if $(u_{i_1}, \ldots, u_{i_n}) \in R$,
- $\neg R(x_{i_1}, \ldots, x_{i_n})$ if $(u_{i_1}, \ldots, u_{i_n}) \notin R$, and
- $\deg_n(x_i)$ if the degree of u_i in $G_{\mathcal{A}}$ is precisely n.

Finally we quantify the variables $x_2, \ldots, x_m$ existentially. Let $\phi(x_1)$ be the resulting formula. It is easy to see that $\mathcal{A} \models \phi(v)$ if and only if $(N_{\mathcal{A}}(r,u), u) \cong (N_{\mathcal{A}}(r,v), v)$. Only the use of the formulas $\deg_n(x_i)$ needs some explanation. If we would omit these formulas, then $\mathcal{A} \models \phi(v)$ would only express that $(N_{\mathcal{A}}(r,u), u)$ is isomorphic to some induced substructure of $(N_{\mathcal{A}}(r,v), v)$ (with u mapped to v). But by fixing the degree of every x_i we exclude the possibility that there exists $y \in S_{\mathcal{A}}(r, x_1)$ with $y \neq x_i$ for all $1 \leq i \leq m$.[2]

Now the automaton $A_{r,u}$ is obtained by translating the formula $\phi(x_1)$ into an automaton using the standard construction for automatic structures, see e.g. [15]: each of the predicates listed above can be translated into an automaton of fixed size (recall that $\deg_n$ is a formula of fixed size). Since we have $2^{O(r)}$ such predicates, their conjunction can be described by a product automaton of size $\exp(2, O(r))$ working on $2^{O(r)}$ tracks (one for each variable x_i). Finally, the existential quantification over the variables $x_2, \ldots, x_m$ means that we have to project this automaton onto the track corresponding to the variable x_1. The resulting automaton is $A_{r,u}$, it still has $\exp(2, O(r))$ states and only one track. This proves Claim 2.

For the next claim we define the norm of an element $u \in L$ as its length $|u|$.

Claim 3. $\mathcal{A}$ is H-bounded by a function H satisfying $H(j,d) \in \exp(3, O(d))$ for all $j \leq d \in \mathbb{N}$.

Proof of Claim 3. By Claim 2, the size of the automaton $A_{r,u}$ is bounded by $\exp(2, c \cdot r)$, where c is some fixed constant. Define the function H by

$$H(j,d) = H(j-1,d) + 2 \cdot \gamma \cdot \exp(2, c \cdot 7^{d-j}),$$

where γ is the constant from Claim 1 and $H(0,d)$ is set to 0. Note that $H(d,d) \in \exp(3, O(d))$. Now let $1 \leq j \leq d$ and $\widetilde{u} = (u_1, \ldots, u_{j-1}) \in L^{j-1}$ with $|u_i| \leq H(i,d)$. Let furthermore $u \in L$ with $|u| > H(j,d)$. Thus, $|u| - |u_i| > 2 \cdot \gamma \cdot \exp(2, c \cdot 7^{d-j})$ for every $1 \leq i \leq j-1$, which by Claim 1 implies that the distance between u and every u_i in the Gaifman-graph is larger than $2 \cdot \exp(2, c \cdot 7^{d-j})$.

[2] The standard solution of this problem is to say that there does not exist $y \notin \{x_1, \ldots, x_m\}$ which is in $G_{\mathcal{A}}$ adjacent to some x_i with $d_{\mathcal{A}}(x_1, x_i) \leq r-1$, see e.g. the proof of [22, Corollary 4.9]. But this would introduce a quantifier alternation that we want to avoid.

Thus, the spheres $S_{\mathcal{A}}(7^{d-j}, \widetilde{u})$ and $S_{\mathcal{A}}(7^{d-j}, u)$ are certainly disjoint and there is no edge in $G_{\mathcal{A}}$ between these two spheres.

Now consider the automaton $A_{7^{d-j},u}$ from Claim 2. It has at most $\exp(2, c \cdot 7^{d-j})$ states. Since u is accepted by $A_{7^{d-j},u}$, it accepts a word of length larger than $H(j,d) = H(j-1,d) + 2 \cdot \gamma \cdot \exp(2, c \cdot 7^{d-j})$. Thus, a simple pumping argument shows that $A_{7^{d-j},u}$ also accepts a word $u_j \in L$ with

$$H(j-1,d) + \gamma \cdot \exp(2, c \cdot 7^{d-j}) \leq |u_j| \leq H(j-1,d) + 2 \cdot \gamma \cdot \exp(2, c \cdot 7^{d-j}) = H(j,d)$$

(note that $\gamma \geq 1$). Since $|u_j| \geq H(j-1,d) + \gamma \cdot \exp(2, c \cdot 7^{d-j})$, Claim 1 implies that the distance between u_j and u_i ($1 \leq i < j$) in the Gaifman-graph is at least $\exp(2, c \cdot 7^{d-j})$. Thus, also the spheres $S_{\mathcal{A}}(7^{d-j}, \widetilde{u})$ and $S_{\mathcal{A}}(7^{d-j}, u_j)$ are disjoint and there is no edge in $G_{\mathcal{A}}$ between these two spheres. Finally, since by definition of the automaton $A_{7^{d-j},u}$ we have $(N_{\mathcal{A}}(7^{d-j}, u), u) \cong (N_{\mathcal{A}}(7^{d-j}, u_j), u_j)$, we obtain $(N_{\mathcal{A}}(7^{d-j}, \widetilde{u}, u), \widetilde{u}, u) \cong (N_{\mathcal{A}}(7^{d-j}, \widetilde{u}, u_j), \widetilde{u}, u_j)$. Thus, $\mathcal{A}$ is H-bounded.

Now we can finish the proof of the theorem. Let

$$\varphi \equiv Q_1 x_1 Q_2 x_2 \cdots Q_d x_d : \psi(x_1, \ldots, x_d)$$

be a first-order sentence over the signature of $\mathcal{A}$ with d quantifiers $Q_i \in \{\exists, \forall\}$. Then, by Corollary 1, $\mathcal{A} \models \varphi$ if and only if

$$\mathcal{A} \models Q_1 x_1 \leq H(1,d)\, Q_2 x_2 \leq H(2,d) \cdots Q_d x_d \leq H(d,d) : \psi(x_1, \ldots, x_d). \quad (3)$$

Since $H(i,d) \in \exp(3, O(|\varphi|))$, this implies the statement of the theorem: In order to verify (3), we guess (either in an existential or a universal state) words $u_i \in L$ with $|u_i| \leq H(i,d)$. Every quantifier alternation leads to one alternation in our alternating Turing-machine. After having guessed every word u_i, we verify whether $\mathcal{A} \models \psi(u_1, \ldots, u_d)$ by running the automata given by the automatic presentation of $\mathcal{A}$. This needs deterministic triply exponential time. This concludes the proof. $\square$

Remark 1. The proof of Theorem 3 shows also another result. Assume that the premises of Theorem 3 are satisfied. If moreover the Gaifman-graph $G_{\mathcal{A}}$ has polynomial growth, i.e., for every $u \in L$, the size of the r-sphere $S_{\mathcal{A}}(r, u)$ is bounded by $r^{O(1)}$, then the size of the automaton $A_{r,u}$ from Claim 2 is bounded by $2^{(n^{O(1)})}$. It follows that $\mathrm{FOTh}(\mathcal{A})$ can be decided in $\mathrm{ATIME}(O(n), \exp(2, n^{O(1)}))$.

Remark 2. Theorem 3 can be easily generalized to a larger class of automatic structures: By Proposition 1, the class of automatic structures is closed under first-order interpretations (see [3] for the definition). Moreover, it is easy to see that a first-order interpretation between two structures leads to a polynomial time reduction between the corresponding first-order theories. Thus, every automatic structure that is first-order interpretable in an automatic structure of bounded degree has a first-order theory in $\mathrm{ATIME}(O(n), \exp(3, O(n)))$. Moreover, the resulting class of automatic structures strictly contains the class of automatic structures of bounded degree.

5 The method of Compton and Henson

In order to prove lower bounds for theories of automatic structures of bounded degree, we will use a method of Compton and Henson, which will be introduced in this section.

For every $i \geq 0$ let $\mathcal{C}_i$ be a class of structures over some signature $(R_j)_{j\in J}$, which is the same for all structures in $\bigcup_{i\geq 0} \mathcal{C}_i$. Assume that R_j has arity n_j. Let furthermore $\mathcal{A}$ be an additional structure with universe A. We say that $(\mathcal{C}_i)_{i\geq 0}$ has a *monadic interpretation* in the structure $\mathcal{A}$ [6] if for every $i \geq 0$ there exist formulas

$$\phi_i(x,r),\ (\psi_{i,j}(x_1,\ldots,x_{n_j},r))_{j\in J},\ \mu_i(x,r,s) \tag{4}$$

over the signature of $\mathcal{A}$ such that for every structure $\mathcal{B} \in \mathcal{C}_i$ there exists $a \in A$ with:

- $\mathcal{B}$ is isomorphic to the structure $(\phi_i(x,a)^{\mathcal{A}}, (\psi_{i,j}(x_1,\ldots,x_{n_j},a)^{\mathcal{A}})_{j\in J})$,
- $\mu_i(x,a,b)^{\mathcal{A}}$ is a subset of $\phi_i(x,a)^{\mathcal{A}}$ for every $b \in A$, and moreover every subset of $\phi_i(x,a)^{\mathcal{A}}$ is of the form $\mu_i(x,a,b)^{\mathcal{A}}$ for some $b \in A$.

Thus, by varying the parameter r in (4), we obtain all structures from $\mathcal{C}_i$. In [6] it is also allowed to use a sequence $r_1,\ldots,r_k$ of parameters instead of a single parameter r. We will not need this more general notion of monadic interpretations.

In order to derive complexity lower bounds using monadic interpretations, one has to require that given $i \geq 0$ in unary notation (i.e., $\$^i$), the formulas in (4) can be computed efficiently. Following [6], we require that these formulas are *reset log-lin computable* from $\$^i$. This means that there exists a deterministic Turing-machine operating in linear time and logarithmic working space that computes (4) from $\$^i$. Moreover the input-head always moves one cell to the right except for k transitions (where k is some fixed constant), where the input-head is reset to the left end of the input. This technical extra condition was introduced in [6] in order to obtain a transitive notion of reducibility.[3] In the following we will always restrict implicitly to reset log-lin computable functions in the context of monadic interpretations. The following theorem was shown in [6, Thm. 7.2].

Theorem 4 (cf. [6]). *Let $T(n)$ be a time resource bound such that for some d between 0 and 1, $T(dn) \in o(T(n))$. Let $\mathcal{C}_n$ be the class of all structures of the form $(\{0,\ldots,m\},\mathrm{plus})$ with $m < T(n)$ and $\mathrm{plus}(x,y,z)$ if and only if $x+y=z$. If there is a monadic interpretation of $(\mathcal{C}_n)_{n\geq 0}$ in a structure $\mathcal{A}$, then for some constant c, $\mathrm{ATIME}(cn, T(cn))$ is a hereditary lower bound for $\mathrm{FOTh}(\mathcal{A})$.*[4]

[3] Reset log-lin reductions should not be confused with the log-lin reductions from [21], where it is only assumed that the output length is linearly bounded in the input length.

[4] From the proofs in [6] it is easy to see that this statement is also true if $\mathcal{C}_n$ is the singleton class $\{(\{0,\ldots,T(n)-1\},\mathrm{plus})\}$.

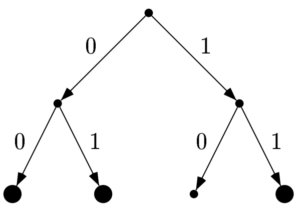

Fig. 1. A tree of height 2 with marked leafs

6 A lower bound

A binary tree of height n with marked leafs is a structure of the form

$$(\{0,1\}^{\leq n}, s_0, s_1, P),$$

where $s_i = \{(w, wi) \mid w \in \{0,1\}^*, |w| < n\}$ and $P \subseteq \{0,1\}^n$ is an additional unary predicate on the leafs. Let $\mathcal{T}_n$ be the set of all these structures and let $\mathcal{T} = \bigcup_{n \geq 0} \mathcal{T}_n$. Figure 1 shows a member of $\mathcal{T}_2$, where the leafs 00, 01, and 11 are marked.

Binary trees with additional unary predicates were used in [10] in order to derive lower bounds on the parametrized complexity of first-order model checking. Here we will use these trees in connection with the method of Compton and Henson from the preceding section. First we have to prove the following lemma:

Lemma 1. *There exists an automatic structure $\mathcal{A} = (A, s_0, s_1, P)$ with $s_i \subseteq A \times A$ and $P \subseteq A$ such that*

- *every connected component of $\mathcal{A}$ is isomorphic to a structure from $\mathcal{T}$, and*
- *every structure from $\mathcal{T}$ is isomorphic to a connected component of $\mathcal{A}$.*

Proof. Let $\Sigma = \{0, 1, \#, a, a', b, b'\}$ and let $A = (\{a, a', b, b'\}^*\{0,1\}^*\#)^*$, which is regular. Let $s_0 \subseteq A \times A$ contain all pairs of the form

$$(u_1\alpha_1 v_1\#u_2\alpha_2 v_2 \cdots \#u_n\alpha_n v_n\#,\ u_1\beta_1 v_1\#u_2\beta_2 v_2 \cdots \#u_n\beta_n v_n\#)$$

such that

- $u_i \in \{a, a', b, b'\}^*$, $\alpha_i \in \{0,1\}$, $\beta_i \in \{a, a', b, b'\}$, $v_i \in \{0,1\}^*$, and
- if $\alpha_i = 0$ then $\beta_i = a$, and if $\alpha_i = 1$ then $\beta_i = b'$.

This relation is clearly automatic. The relation $s_1 \subseteq A \times A$ is defined analogously, we only replace the second condition above by $\beta_i = b$ if $\alpha_i = 1$ and $\beta_i = a'$ if $\alpha_i = 0$. Finally define the regular language $P \subseteq A$ by

$$P = \{w_1\#w_2\# \cdots w_n\# \in A \mid w_i \in \{a, b\}^* \text{ for some } i\}.$$

This finishes the definition of the automatic structure $\mathcal{A} = (A, s_0, s_1, P)$. It is easy to see that $\mathcal{A}$ has indeed the properties stated in the lemma. In Figure 2, it is shown, how the tree from Figure 1 is generated. Marked leafs are

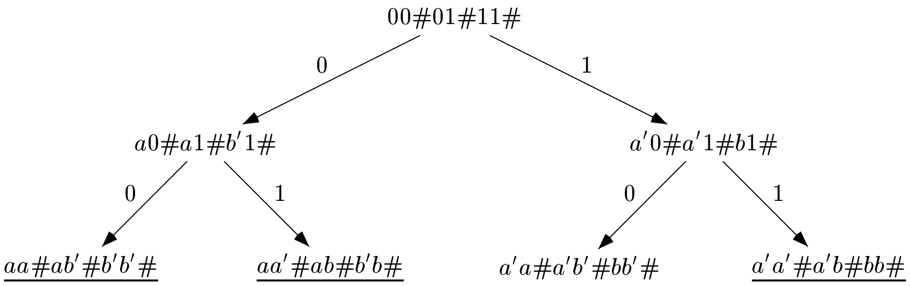

Fig. 2. A connected component from the structure $\mathcal{A}$ in Lemma 1

underlined. Note that the same tree is for instance also rooted at words from $\{a,b\}^*00\#\{a,b\}^*01\#\{a,b\}^*11\#(\{a,a',b,b'\}^*\{0,1\}^{\geq 3}\#)^*$.

$\square$

Lemma 2. *Let $\mathcal{A}$ be the structure from Lemma 1. Let $\mathcal{C}_n$ be the class of all structures of the form $(\{0,\dots,m\},\text{plus})$ with $m < 2^{2^n}$. Then there exists a monadic interpretation of $(\mathcal{C}_n)_{n\geq 0}$ in the structure $\mathcal{A}$.*

Proof. Let $\mathcal{A} = (A, s_0, s_1, P)$ be the structure from Lemma 1. Given $a, b \in A$ we say that b is a successor of a or a is a predecessor of b if there exists a directed path in the relation $s_0 \cup s_1$ from a to b. This directed path defines a word over $\{0,1\}$ in the canonical way. If, e.g., $s_0(a,c)$ and $s_1(c,b)$, then the path from a to b defines the word 01.

Our aim is to construct formulas

$$\phi_n(x,r),\ \text{plus}_n(x,y,z,r),\ \mu_n(x,r,s)$$

that are reset log-lin computable from $\$^n$ and that witness a monadic interpretation of $(\mathcal{C}_n)_{n\geq 0}$ in the structure $\mathcal{A}$. First, we define a few auxiliary formulas that define relations in the structure $\mathcal{A}$. Let us fix $n \geq 0$. For every $0 \leq i \leq n$, the formula $\pi_i(x_0,x_1,y_0,y_1)$ expresses that x_0 is a predecessor of x_1, y_0 is a predecessor of y_1, and the unique path leading in $\mathcal{A}$ from x_0 to x_1 has length at most 2^i and is labeled with the same word over $\{0,1\}$ as the unique path leading in $\mathcal{A}$ from y_0 to y_1:

$$\begin{aligned}
&\pi_0(x_0,x_1,y_0,y_1) \equiv \\
&\quad (x_0 = x_1 \wedge y_0 = y_1) \vee (s_0(x_0,x_1) \wedge s_0(y_0,y_1)) \vee (s_1(x_0,x_1) \wedge s_1(y_0,y_1)) \\
&\pi_{i+1}(x_0,x_1,y_0,y_1) \equiv \\
&\quad \exists x_2\, \exists y_2\, \forall x\, \forall x'\, \forall y\, \forall y' \\
&\qquad \left\{ \begin{pmatrix} (x = x_0 \wedge x' = x_2 \wedge y = y_0 \wedge y' = y_2) \vee \\ (x = x_2 \wedge x' = x_1 \wedge y = y_2 \wedge y' = y_1) \end{pmatrix} \to \pi_i(x,x',y,y') \right\}
\end{aligned}$$

Here we use the usual trick for replacing two occurrences of π_i in the definition of π_{i+1} by a single occurrence of π_i [9], which is necessary in order to obtain formulas of linear size. It is easy to see that π_i is reset log-lin computable from $\$^i$ (see [6] for a class of reset log-lin computable formula sequences that contains the sequence $(\pi_i)_{i\geq 0}$). In the same way we can construct reset log-lin computable formulas, which express the following:

- $\preceq_i(x,y)$ if and only if x is a predecessor of y and the unique path from x to y has length at most 2^i. Instead of $\preceq_i(x,y)$ we will write $x \preceq_i y$. We write $x \prec_i y$ if $x \preceq_i y$ and $x \neq y$.
- $\mathrm{dist}_i(x_0,x_1,y_0,y_1)$ if and only if $x_0 \prec_i x_1$, $y_0 \prec_i y_1$, and the unique path from x_0 to x_1 has the same length as the unique path from y_0 to y_1. We write $\lambda_i(x,y,z)$ for $\mathrm{dist}_i(x,y,x,z)$.

We will represent an interval $\{0,\ldots,m\}$ with $m < 2^{2^n}$ by the leafs of a binary tree of height $k \leq 2^n$ rooted at the node $r \in A$.[5] The set of these nodes can be defined by the formula

$$\phi_n(x,r) \ \equiv\ r \preceq_n x \ \wedge\ \neg\exists y\,\{s_0(x,y) \vee s_1(x,y)\}$$

(thus, for most $a \in \mathcal{A}$ we have $\phi_n(x,a)^{\mathcal{A}} = \emptyset$). The word from $\{0,1\}^k$ $(k \leq 2^n)$ labeling the path from the root r to a leaf x can be interpreted as the binary coding of x. In order to define addition on these leafs let y be another leaf of the tree rooted at r. Let $u \neq r$ (resp. $v \neq r$) be a node on the unique path from r to x (resp. r to y). Assume that $\lambda_n(r,u,v)$ holds. We first define a formula $\psi_n(u,v,r)$, expressing that adding x and y leads to a carry over from a previous position at the position corresponding to u (and v). For $i \in \{0,1\}$ let $\beta_i(x) \equiv \exists y : s_i(y,x)$. Then we can define $\psi_n(u,v,r)$ as follows:

$$\psi_n(u,v,r) \ \equiv\ \exists p\,\exists q \left\{ \begin{array}{l} \lambda_n(r,p,q) \ \wedge\ p \prec_n u \ \wedge\ q \prec_n v \ \wedge\ \beta_1(p) \ \wedge\ \beta_1(q) \ \wedge \\ \forall s\,\forall t \left\{ \begin{pmatrix} \lambda_n(r,s,t) \ \wedge \\ p \preceq_n s \prec_n u \ \wedge \\ q \preceq_n t \prec_n v \end{pmatrix} \rightarrow (\beta_1(s) \ \vee\ \beta_1(t)) \right\} \end{array} \right\}$$

Using the formula

$$\begin{aligned} \varphi_n(u,v,w,r) \ \equiv\ & \bigvee_{\substack{i,j,k\in\{0,1\} \\ i+j+1\equiv k \bmod 2}} (\beta_i(u) \ \wedge\ \beta_j(v) \ \wedge\ \psi_n(u,v,r) \ \wedge\ \beta_k(w)) \ \vee \\ & \bigvee_{\substack{i,j,k\in\{0,1\} \\ i+j\equiv k \bmod 2}} (\beta_i(u) \ \wedge\ \beta_j(v) \ \wedge\ \neg\psi_n(u,v,r) \ \wedge\ \beta_k(w)) \end{aligned}$$

we can define $\mathrm{plus}_n(x,y,z,r)$ as follows:

$$\begin{aligned} \mathrm{plus}_n(x,y,z,r) \ \equiv\ & \phi_n(x,r) \ \wedge\ \phi_n(y,r) \ \wedge\ \phi_n(z,r) \ \wedge \\ & \forall u\,\forall v\,\forall w \left\{ \begin{pmatrix} \lambda_n(r,u,v) \ \wedge\ \lambda_n(r,v,w) \ \wedge \\ u \preceq_n x \ \wedge\ v \preceq_n y \ \wedge\ w \preceq_n z \end{pmatrix} \rightarrow \varphi_n(u,v,w,r) \right\} \end{aligned}$$

[5] In this way we represent only those intervals whose size is a power of two, which is not crucial, see the footnote in Theorem 4.

Finally, arbitrary subsets of the set of leafs in the tree rooted at r can be defined by varying s in the following formula:

$$\mu_n(x,r,s) \equiv \phi_n(x,r) \wedge \exists y\,\{s \preceq_n y \wedge \pi_n(r,x,s,y) \wedge P(y)\}$$

This formula selects those leafs from the tree rooted at r such that the corresponding leaf in the tree rooted at s satisfies the unary predicate P. $\square$

Lemma 1 and 2 combined with Theorem 4 give us the main result of this section:

Theorem 5. *There exists an automatic structure $\mathcal{A}$ of bounded degree such that for some constant c,* $\mathrm{ATIME}(cn, \exp(2,cn))$ *is a hereditary lower bound for* $\mathrm{FOTh}(\mathcal{A})$.

7 Tree automatic structures

Tree automatic structures were introduced in [2], they generalize automatic structures. Let Γ be a finite alphabet. A *finite binary tree* over Γ is a mapping $t : \mathrm{dom}(t) \to \Gamma$, where $\mathrm{dom}(t) \subseteq \{0,1\}^*$ is finite and satisfies the following closure condition for all $w \in \{0,1\}^*$ and $i \in \{0,1\}$: if $wi \in \mathrm{dom}(t)$, then also $w \in \mathrm{dom}(t)$ and $wj \in \mathrm{dom}(t)$ for all $j \in \{0,1\}$. Those $w \in \mathrm{dom}(t)$ such that $w0 \notin \mathrm{dom}(t)$ (and hence also $w1 \notin \mathrm{dom}(t)$) are called the *leafs* of t. With T_Γ we denote the set of all finite binary trees over Γ. We define the *height* of the tree t by $\mathrm{height}(t) = \max\{|w| \mid w \in \mathrm{dom}(t)\}$. A *tree automaton over Γ* is a tuple $A = (Q, \delta, I, F)$, where Q is the finite set of states, $I \subseteq Q$ (resp. $F \subseteq Q$) is the set of initial (resp. final) states, and $\delta \subseteq Q \times Q \times \Gamma \times Q$. A *successful run* of A on a tree t is a mapping $\rho : \mathrm{dom}(t) \to Q$ such that: (i) $\rho(w) \in I$ if w is a leaf of t, (ii) $\rho(\epsilon) \in F$, and (iii) $(\rho(w0), \rho(w1), t(w), \rho(w)) \in \delta$ if $w \in \mathrm{dom}(t)$ is not a leaf. With $T(A)$ we denote the set of all finite binary trees t such that there exists a successful run of A on t. A set $L \subseteq T_\Gamma$ is called *recognizable* if there exists a finite tree automaton A with $L = T(A)$. Recognizable tree languages allow similar pumping arguments as regular word languages. More precisely, if A is a finite tree automaton with n states and $T(A) \neq \emptyset$, then $T(A)$ contains a tree of height at most n.

Let $t_1, \ldots, t_n \in T_\Gamma$. We define the convolution $t = t_1 \otimes \cdots \otimes t_n$, which is a finite binary tree over $\prod_{i=1}^n (\Gamma \cup \{\#\})$, as follows: $\mathrm{dom}(t) = \bigcup_{i=1}^n \mathrm{dom}(t_i)$ and for all $w \in \bigcup_{i=1}^n \mathrm{dom}(t_i)$ we define $t(w) = (a_1, \ldots, a_n)$, where $a_i = t_i(w)$ if $w \in \mathrm{dom}(t_i)$ and $a_i = \#$ otherwise. An n-ary relation R over T_Γ is called tree-automatic if the language $\{t_1 \otimes \cdots \otimes t_n \mid (t_1, \ldots, t_n) \in R\}$ is recognizable. Using this definition we can define the notion of a *tree automatic presentation* analogously to the word case in Section 2: A tree automatic presentation of the structure $\mathcal{A} = (A, (R_i)_{i \in J})$, where $R_i \subseteq A^{n_i}$, is a tuple (Γ, L, h) such that

- Γ is a finite alphabet,
- $L \subseteq T_\Gamma$ is recognizable,
- $h : L \to A$ is a surjective function,

– the relation $\{(u, v) \in L \times L \mid h(u) = h(v)\}$ is tree automatic, and
– the relation $\{(u_1, \ldots, u_{n_i}) \in L^{n_i} \mid (h(u_1), \ldots, h(u_{n_i})) \in R_i\}$ is tree automatic for every $i \in J$.

We say that $\mathcal{A}$ is *tree automatic* if there exists a tree automatic presentation for $\mathcal{A}$. An example of a tree automatic structure, which is not automatic is $(\mathbb{N}, \cdot)$, i.e., the natural numbers with multiplication [2].

Many results for automatic structures carry over to tree automatic structures. For instance the first-order theory of a tree automatic structure is still decidable [2]. Analogously to Theorem 3 we can prove the following result:

Theorem 6. *If $\mathcal{A}$ is a tree automatic structure of bounded degree, then* $\mathrm{FOTh}(\mathcal{A})$ *can be decided in* $\mathrm{ATIME}(O(n), \exp(4, O(n)))$.

Proof. We copy the proof of Theorem 3. Thus, let (Γ, L, h) be a tree automatic presentation for $\mathcal{A}$, where h can be assumed to be bijective (see [2, Theorem 3.4]). For an element $t \in L$, we define its norm as $\mathrm{height}(t)$. Then, analogously to Claim 1 in the proof of Theorem 3 it follows that if (t, t') is an edge in the Gaifman-graph $G_{\mathcal{A}}$, then $|\mathrm{height}(t) - \mathrm{height}(t')| \leq \gamma$. Then also Claim 2 and 3 from the proof of Theorem 3 carry over easily to the tree automatic case. Thus $\mathcal{A}$ is H bounded by a function H satisfying $H(j, d) \in \exp(3, O(d))$ for all $j \leq d \in \mathbb{N}$. We can conclude as in the word case. The only difference is that a binary tree, whose height is bounded by $\exp(3, O(n))$ needs $\exp(4, O(n))$ many bits for its specification in the worst case. This is the reason for the $\mathrm{ATIME}(O(n), \exp(4, O(n)))$ upper bound in the theorem. □

8 Open problems

Several open problems remain for (tree) automatic structures of bounded degree:

– Does there exist an automatic structure $\mathcal{A}$ of bounded degree such that $\mathrm{ATIME}(O(n), \exp(3, O(n)))$ is a (hereditary) lower bound for $\mathrm{FOTh}(\mathcal{A})$, or is $\mathrm{ATIME}(O(n), \exp(2, O(n)))$ always an upper bound? The same open problem remains for tree automatic structures, there the gap is even larger (between $\mathrm{ATIME}(O(n), \exp(2, O(n)))$ and $\mathrm{ATIME}(O(n), \exp(4, O(n)))$).
– Is there a tree automatic structure of bounded degree, which is not automatic? Without the restriction to structures of bounded degree this is true, see Section 7.

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