On Spectral Lower Bound Arguments for Decision Trees

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A decision tree is a rooted binary tree with inner nodes labeled by Boolean variables and edges and leaves labeled by Boolean constants. A decision tree T defines a Boolean function f_T in a natural way: given an input a, start at the root and follow at each inner node the edge according to the value of the variable that labels this node. The label of the terminal node is $f_T(a)$. The *size* of a decision tree is its number of leaves, the *average depth* is the expectation of the depth of the leaf that is reached by a randomly chosen input. Denote by $\mathsf{DT}(f)$ and $\overline{\mathsf{depth}}(f)$ the minimal size and the minimal average depth, respectively, of decision trees that compute a given Boolean function $f : \{0,1\}^n \to \{+1,-1\}$.

We give an new proof for the following lower bounds due to Brandman, Orlitsky, and Hennessy ([1]):

$$\mathsf{DT}(f) \geq 2^n \sum_{S\subseteq[n]} 2^{|S|} \cdot \hat{f}(S)^2, \overline{\mathsf{depth}}(f) \geq \sum_{S\subseteq[n]} |S| \cdot \hat{f}(S)^2.$$

Here the $\hat{f}(S)$ are defined by the decomposition $f = \sum_{S\subseteq[n]} \hat{f}(S)\chi_S$, where χ_S is the ± 1-valued parity of the variables $x_i, i \in S$.

Our argument is extremely easy and it is easily extensible to spectral decompositions with respect to bases different from the standard Fourier basis $\chi_S, S \subseteq [n]$. In case f is symmetric, for instance, it is natural to include symmetric functions as basis functions. Representing symmetric functions as polynomials in a single real variable, we can take orthogonal polynomials as basis functions. An example are the KRAWTCHOUK-polynomials, that have many applications in the theory of error correcting codes (cf. [2]). This leads to modified lower bound formulas involving only a linear number of spectral coefficients.

Literatur

[1] Y. Brandman, A. Orlitsky, and J. Hennessy. A spectral lower bound technique for the size of decision trees and two-level AND/OR circuits. *IEEE Transactions on Computers*, 39:282–287, 1990.

[2] J. MacWilliams and N. Sloane. *The Theory of Error-Correcting Codes.* Amsterdam: North-Holland Publishing Co., 1977.