Geodesic rewriting systems and pregroups*

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Abstract

In this paper we study rewriting systems for groups and monoids, focusing on situations where finite convergent systems may be difficult to find or do not exist. We consider systems which have no length increasing rules and are confluent and then systems in which the length reducing rules lead to geodesics. Combining these properties we arrive at our main object of study which we call geodesically perfect rewriting systems. We show that these are well-behaved and convenient to use, and give several examples of classes of groups for which they can be constructed from natural presentations. We describe a Knuth-Bendix completion process to construct such systems, show how they may be found with the help of Stallings' pregroups and conversely may be used to construct such pregroups.

Contents

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1 Introduction

A presentation of a group or monoid may be thought of as a rewriting system which, in certain cases may give rise to algorithms for solving classical algorithmic problems. For example if the rewriting system is finite and convergent (that is confluent and terminating) then it can be used to solve the word problem and to find normal forms for elements of the group. This is one reason for the importance of convergent rewriting systems in group theory. However there are many groups for which the natural presentations do not give rise to convergent rewriting systems, but which are none the less well behaved, algorithmically tractable groups. In this paper we investigate properties of rewriting systems, which are not in general finite or terminating, but which all the same give algorithms for such tasks as solving the word problem, computation of normal forms or computation of geodesic representatives of group elements. We contend that the resulting algorithms are often more convenient and practical than those arising from more conventional finite convergent systems.

Rewriting methods in algebra have a very long and rich history. In groups and semigroups they are usually related to the word problem and take their roots in the ground breaking works of Dehn and Thue (not to mention the classical Euclidean and Gaussian elimination algorithms!). Several famous algorithms in group theory are in fact particular types of string rewriting processes: the Nielsen method in free groups, Hall collection in nilpotent and polycyclic groups, the Dehn algorithm in small cancellation and hyperbolic groups, Tits rewriting in Coxeter groups, convergent rewriting systems for finite groups, and so on. In rings and algebras rewriting methods appear as a major tool in computing normal forms of elements [40, 45, 10], in solving the word and ideal-membership problems. These techniques emerged up independently in various branches of algebra at different times and under different names (the diamond lemma, Gröbner or Shirshov bases, Buchberger's algorithm and S-polynomials, for instance). They have gained prominence with the progress of practical computing, as real applications have become available. Notably, crucial developments in methods of computational algebra originated in commutative algebra and algebraic geometry, with Buchberger's celebrated algorithm and related computational techniques, which revolutionised the whole area of applications. We refer to [11], and the references therein, for more details.

From the theoretical view point the main shift in the paradigm came with the seminal paper of Knuth and Bendix [33]. In this paper they introduced a process, now known as the Knuth-Bendix (KB) procedure, which unified the field of rewriting techniques in (universal) algebra. This KB procedure gives a solid theoretical basis for practical implementations, even the procedure itself may lead to non-optimal algorithms for solving word problems.

Roughly speaking a KB procedure takes as input a finite system of identities (between terms) and a computable (term) ordering such that the identities can be read as a finite set of directed rewrite rules. Using the crucial concept of *critical pairs* the procedure adds in each round more and more rules, and it stops only if the system is *completed*. Thus the KB procedure attempts to construct an equivalent convergent (term) rewriting system: which in particular allows unique normal forms to be found by a simple strategy. In case of termination we obtain a solvable word problem.

In the case of commutative algebra this concept can be viewed as Buchberger's algorithm and termination is guaranteed. In case of algebraic structures like groups or monoids we have a special case of term rewriting systems since the rewriting process is based on strings. (Formally, monoid generators are read as unary function symbols, and the neutral element is read a constant.)

As has been mentioned above, the history of rewriting systems in monoids and groups is about one hundred years old, with the main focus on convergent rewriting systems and algorithms for computing normal forms. Any presentation $M = \langle \Gamma \mid \ell_i = r_i (i \in I) \rangle$ of a monoid M gives a rewriting system $S = \{\ell_i \to r_i, i \in I\}$ which defines M via the congruence relation it generates on the free monoid Γ^*. Every rule $\ell \to r \in S$ allows one to rewrite a word $u\ell v$ into the word urv and this gives a (non-deterministic) word rewriting procedure associated with S. If the system S is convergent (see Section 2) then this rewriting system describes a deterministic algorithm which computes the normal forms of elements, thus solving the word problem in the monoid M. This yields the major interest in finite convergent systems. Many groups are known to allow finite convergent systems (for example Coxeter groups, polycyclic groups, some small cancellation groups: see books [30, 46, 34] for more examples and details). The primary task here is to find a finite convergent system for a given finitely presented monoid, assuming that such a system exists. In principle, the KB procedure performs this task. However, several obstacles may present themselves. By design, to start the KB procedure one has to fix in advance an ordering on Γ^*, with particular properties, as described in Section 2.3. This may seem like a minor hurdle, but the difficulty is that, even for well understood groups, with two orderings which look very much alike, it may happen that using the first the KB process halts and outputs a convergent system while with respect to the the second there exists no finite convergent system: see Example 2.5 below. Furthermore, the existence of a finite convergent system also depends on a choice of the set of generators of the group. This means that for KB to succeed one has to make a the right choice of a set of generators Γ and of an ordering on Γ^*. In fact [42] in general the problem of whether or not a given finitely presented group can be defined by a finite convergent rewriting system is undecidable. In addition, even when restricted to instances where the generators and the order have been chosen so that the KB process will halt giving a finite convergent rewriting system, there may be no be effectively computable upper bound on the running time of the KB procedure. To make things even more interesting, having a finite convergent rewriting system S does not guarantee a fast solution of the word problem in the monoid M (see Section 2.3). All these results show that the KB process for finite convergent systems, while being an important theoretical tool, is not a panacea for problems in computational algebra.

As a first step towards resolving some of these difficulties we consider, in Section 4, the class of preperfect rewriting systems: that is those which are confluent and have no length increasing rules. These restrictions are enough to allow solution of the word problem and to find geodesic representatives and, as examples show, such systems are common in geometric group theory. In fact in Section 7 we describe preperfect rewriting systems for Coxeter groups, graph groups, HNN-extensions and free products with amalgamation. One disadvantage of these systems is that it is undecidable whether are a finite rewriting system is preperfect or not [38] (see Theorem 4.6).

Another desirable property of rewriting systems is that they should be geodesic; meaning that shortest representatives of elements can be found by applying only the length reducing rules of the system. A group defined by a finite geodesic rewriting system has solvable word problem and in [24] these groups are characterised as the finitely generated virtually free groups. However, as we show in Section 5.1, the question of a whether or not a finite rewriting system is geodesic is undecidable.

Combining properties of preperfect and geodesic rewriting systems we arrive at geodesically perfect rewriting systems (defined in Section 5.2). These were first investigated by Nivat and Benois [41] where they were called *quasi-parfaites*. Elsewhere these rewriting systems are also known as *almost confluent*, see e.g. [6] but here we prefer the notation geodesically perfect since these systems are designed to deal with geodesics in groups and monoids. In [41], it was shown that the property of being geodesically perfect is decidable for finite systems. This leads to a new Knuth-Bendix completion procedure for constructing geodesically perfect systems as we explain below. One advantage of this KB process is that it requires no choice of ordering, using only the partial order given by word length in Γ^*.

Among the examples of Section 7 are rewriting systems for amalgamated products and HNN-extensions. As several several important frameworks have been developed to unify the studies of such groups (Bass-Serre Theory, pregroups and relatively hyperbolic groups,for example) it is natural to look for a unified theory of rewriting systems covering HNN-extensions and amalgamated products. In this paper, following Stallings [49, 50], we approach this unification question from a combinatorial view-point via pregroups and their universal groups: which seem to lend themselves naturally to algorithmic and model theoretic problems. Intuitively, a pregroup can be viewed as a "partial group", that is, a set P with a partial (not everywhere defined) multiplication $m : P \times P \to P$, or a piece of the multiplication table of some group, that satisfies some particular axioms. In this case the universal group $U(P)$ can be described as the group defined by the presentation with a generating set P and a set of relations $m(x, y) = z$ for all $x, y \in P$ such that $m(x, y)$ is defined and equal to z. On the other hand, Stallings proved that $U(P)$ can be realized constructively as the set of all P-reduced forms (reduced sequences of elements of P) modulo a suitable equivalence relation and a naturally defined multiplication. We discuss these definitions in detail in Section 8.

In Section 8.1 we show how the existence of a pregroup allows us to construct a preperfect rewriting system for the universal group. Moreover, we show in Theorem 8.4 that this system is geodesically perfect. In this way pregroups may play a role in clarifying completion procedures of KB type. In particular, completing a given presentation (in terms of generators and relators) of a group G to a larger presentation, which is a pregroup, amounts to a construction of a geodesically perfect rewriting system for G.

As an application of these results we obtain a slight strengthening of the result of [24]. It is known that a group G is virtually free if and only if $G = U(P)$ for a finite pregroup P [44] and combining this result with Theorem 8.4 we see that a group is finitely generated, virtually free if and only if it is defined by a geodesically perfect rewriting system (Corollary 8.7).

2 Rewriting techniques

2.1 Basics

In this section we recall the basic concepts from string rewriting. We use rewriting techniques as a tool to prove that certain constructions have the expected properties.

A *rewriting relation* over a set X is a binary relation $\Longrightarrow \subseteq X \times X$. We denote by $\overset{*}{\Longrightarrow}$ the reflexive and transitive closure of $\Longrightarrow$, by $\Longleftrightarrow$ its symmetric closure and by $\overset{*}{\Longleftrightarrow}$ its symmetric, reflexive, and transitive closure. We also write $y \Longleftarrow x$ whenever $x \Longrightarrow y$, and we write $x \overset{\leq k}{\Longrightarrow} y$ whenever we can reach y in at most k steps from x.

Definition 2.1. *The relation $\Longrightarrow \subseteq X \times X$ is called:*

i) strongly confluent, *if $y \Longleftarrow x \Longrightarrow z$ implies $y \overset{\leq 1}{\Longrightarrow} w \overset{\leq 1}{\Longleftarrow} z$ for some w,*

ii) confluent, *if $y \overset{*}{\Longleftarrow} x \overset{*}{\Longrightarrow} z$ implies $y \overset{*}{\Longrightarrow} w \overset{*}{\Longleftarrow} z$ for some w,*

iii) Church-Rosser, *if $y \overset{*}{\Longleftrightarrow} z$ implies $y \overset{*}{\Longrightarrow} w \overset{*}{\Longleftarrow} z$ for some w,*

iv) locally confluent, *if $y \Longleftarrow x \Longrightarrow z$ implies $y \overset{*}{\Longrightarrow} w \overset{*}{\Longleftarrow} z$ for some w,*

The following facts are well-known and can be found in several text books (see for example, [6, 31]).

1) Strong confluence implies confluence.

2) Confluence is equivalent to Church-Rosser.

3) Confluence implies local confluence, but the converse is false, in general.

2.2 Rewriting in monoids

Rewriting systems over monoids (and in particular over groups) play an important part in algebra. Let M be a monoid. A *rewriting system* over M is a binary relation $S \subseteq M \times M$. It defines the rewriting relation $\underset{S}{\Longrightarrow} \subseteq M \times M$ such that

$$x \underset{S}{\Longrightarrow} y \text{ if and only if } x = p\ell q,\ y = prq \text{ for some } (\ell, r) \in S.$$

The relation $\overset{*}{\underset{S}{\Longleftrightarrow}} \subseteq M \times M$ is a congruence on M, hence the quotient set $M/\overset{*}{\underset{S}{\Longleftrightarrow}}$ forms a monoid with respect to the multiplication induced from M. We denote it by $M/\{\, \ell = r \mid (\ell, r) \in S \,\}$ or, simply by M/S. Two rewriting systems S and T over a monoid M are termed *equivalent* if $\overset{*}{\underset{S}{\Longleftrightarrow}} = \overset{*}{\underset{T}{\Longleftrightarrow}}$, i.e., $M_S = M_T$.

We say that a rewriting system S is strongly confluent (or confluent, etc) if the relation $\underset{S}{\Longrightarrow}$ has the corresponding property. Instead of $(\ell, r) \in S$ we also write $\ell \longrightarrow r \in S$ and $\ell \longleftrightarrow r \in S$ in order to indicate that both (ℓ, r) and (r, ℓ) are in S.

We say that a word w is *S-irreducible* (sometimes we omit S here), if no left-hand side ℓ of S occurs in w as a factor. Thus, if w is irreducible, then $w \overset{*}{\underset{S}{\Longrightarrow}} w'$ implies $w = w'$. The set of all irreducible words is denoted by $\mathrm{IRR}(S)$.

In order to compute with monoids (in particular, groups) we usually specify a choice of monoid generators Γ, sometimes called an *alphabet*. For groups we often assume that Γ is closed under inversion, so $\Gamma = \Sigma \cup \Sigma^{-1}$ where Σ is a set of group generators. For an alphabet Γ we denote by Γ^* the free monoid with basis Γ. Throughout, 1 denotes the neutral element in monoids or groups. In particular, 1 is also used to denote the *empty word* in a free monoid Γ^*. If we can write $w = xuy$, then we say that u is a factor of w. For free monoids a factor is sometimes also called a *subword*, but this might lead to confusion because other authors understand b a subword simply a subsequence or *scattered subword*.

Rewriting systems S over a free monoid Γ^* are sometimes called *string rewriting systems* or *semi-Thue systems*. In this case the quotient Γ^*/S has the standard monoid presentation $\langle \Gamma \mid \{\, \ell = r \mid (\ell, r) \in S \,\} \rangle$. We say that a string rewriting system S defines a monoid M if Γ^*/S is isomorphic to M. In addition, if P is a property of rewriting systems (Church-Rosser, strongly confluent, confluent, etc.) we say that a monoid M has a P-presentation if it can be defined by a system with property P.

For groups two types of presentations via generators and relators arise: monoid presentations, described above, and group presentations, typical in combinatorial group theory and topology. More precisely, we say that $G = \Gamma^*/S$ is a monoid presentation of a group G if the alphabet Γ is of the form $\Gamma = \Sigma \cup \Sigma^{-1}$, where Σ is a set of group generators, and $\Sigma^{-1} = \{\sigma^{-1} \mid \sigma \in \Sigma\}$ is the set of formal inverses of Σ (in which case Γ^* is a the free monoid with an involution $\sigma \to \sigma^{-1}$). Given a group presentation $\langle X \mid R \rangle$ of a group G one can easily obtain a monoid presentation of G by adding the formal inverses X^{-1} to the set of generators X of G and the "trivial" relations $xx^{-1} = 1, x^{-1}x = 1, x \in X\}$ to the relators of G. We consider here monoid presentations of groups, except where explicitly indicated otherwise.

2.3 Convergent rewriting systems

In this section we briefly discuss *convergent* (or *complete*) rewriting systems, which play an important role in algebra due to their relation to normal forms.

A relation $\Longrightarrow \subseteq X \times X$ is called *terminating* (or *Noetherian*), if every infinite chain

$$x_0 \overset{*}{\Longrightarrow} x_1 \overset{*}{\Longrightarrow} \cdots x_{i-1} \overset{*}{\Longrightarrow} x_i \overset{*}{\Longrightarrow} \cdots$$

becomes stationary.

There are two typical sources of terminating string rewriting systems $S \subseteq \Gamma^* \times \Gamma^*$. Systems of the first type are *length-reducing*, i.e., for any rule $\ell \to r \in S$ one has $|\ell| > |r|$, where $|x|$ is the length of a word $x \in \Gamma^*$. Systems of the second type are *compatible* with a given *reduction* ordering $\succ$ on Γ^*, which means that if $\ell \to r \in S$ then $\ell \succ r$. Recall that a reduction ordering on Γ^* is a well-ordering preserving left and right multiplication (i.e. if $u \succ v$ then $aub \succ avb$ for any $a, b \in \Gamma^*$). Clearly, such systems are terminating. In fact, the condition that S is compatible with some partial order, $\succ$, preserving left and right multiplication is just a reformulation of the terminating property. Indeed, if S is terminating then there is a binary relation $\succ_S$ on Γ^* defined by $u \succ_S v$ if and only if $u \overset{*}{\underset{S}{\Longrightarrow}} v$. In this case $\succ_S$ is a partial well-founded ordering (no infinite descending chains), such that $\ell \succ_S r$ for any rule $\ell \to r \in S$. Moreover, the converse is also true. (The condition that $\succ$ is total is not needed here but is required in running the Knuth-Bendix completion procedure, see below).

A relation $\Longrightarrow \subseteq X \times X$ is called *convergent* (or *complete*) if it is locally confluent and terminating. The following properties are crucial. Let S be a convergent rewriting system.

1) S is confluent (see for example, [6, 31]).

2) Every $\overset{*}{\underset{S}{\Longleftrightarrow}}$ equivalence class in Γ^* contains a unique S-reduced word (a word to which no rule from S is applicable).

3) If S is finite then for a given word $w \in \Gamma^*$ one can effectively find its unique S-reduced form (just by subsequently rewriting the word w until the result is S-reduced).

The results above show that if a monoid M has a finite convergent presentation then the word problem in M as well as the problem of finding the normal forms, is decidable. This explains popularity of convergent systems in algebra. There are many examples of groups that have finite convergent presentations: finite groups, polycyclic group, free groups, some geometric groups (see [46, 20, 34] for details)

One of the major results on convergent systems concerns the Knuth-Bendix procedure (KB) (see [6] for general rewriting systems and [46, 20] for groups), which can be stated as follows. Let $\succ$ be a reduction well-ordering on Γ^* and $S \subseteq \Gamma^* \times \Gamma^*$ a finite rewriting system compatible with $\succ$. If there exists a finite convergent rewriting system $T \subseteq \Gamma^* \times \Gamma^*$ compatible with $\succ$ which is equivalent to S, then, in finitely many steps, the Knuth-Bendix procedure KB finds a finite convergent rewriting system $S' \subseteq \Gamma^* \times \Gamma^*$ compatible with $\succ$ which is also equivalent to S.

There are three principle remarks due here.

Remark 2.2. *The time complexity of the word problem in a monoid M_S defined by a finite convergent system S may be of an arbitrarily high complexity [43].*

Remark 2.3. *It may happen that the word problem in a monoid M_S defined by a finite convergent system S is decidable in polynomial time, whereas the complexity of the standard rewriting algorithm that finds the S-reduced forms of words can be of an arbitrarily high complexity [43].*

These remarks show that convergent rewriting systems may not be the best tool to deal with complexity issues related to the word problems and normal forms in monoids.

Remark 2.4. *The Knuth-Bendix procedure really depends on the chosen ordering $\succ$. The following example shows that in a free Abelian group of rank two the KB procedure relative to one length-lexicographic ordering results in a finite convergent presentation, while another length-lexicographic ordering does not allow any finite convergent presentations for the same group.*

Example 2.5 ([21], page 127)**.** *Let G be the free Abelian group given by the following monoid presentation.*

$$\langle x, y, x^{-1}, y^{-1} \mid xy = yx, xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1\rangle.$$

Then the KB procedure with respect to the length-lexicographic ordering induced by the ordering $x < x^{-1} < y < y^{-1}$ of the generators outputs a finite convergent system defining G:

$$xx^{-1} \Longrightarrow 1, x^{-1}x \Longrightarrow 1, yy^{-1} \Longrightarrow 1, y^{-1}y \Longrightarrow 1,$$

$$yx \Longrightarrow xy, y^{-1}x \Longrightarrow xy^{-1}, yx^{-1} \Longrightarrow x^{-1}y, y^{-1}x^{-1} \Longrightarrow x^{-1}y^{-1}.$$

However, there are no finite convergent systems defining G and compatible with the length-lexicographic ordering $x < y < x^{-1} < y^{-1}$.

Therefore, even if a finite convergent presentation for a monoid M exists it might be hard to find it using the Knuth-Bendix procedure. In addition Ó'Dúnlaing [42] has shown that the problem of whether or not a given finitely presented group can be defined by a finite convergent rewriting system is undecidable.

It is not hard to see that all finitely generated commutative monoids have a finite convergent presentation, [13]. However, this is demands enough generators, in general. For example, a free Abelian groups of rank k can be generated as a monoid by an alphabet of size $k+1$, but in order to find a finite convergent system for it we need at least $2k$ generators, see [14]. Another nice example of this kind is the non-commutative semi-direct product of $\mathbb{Z}$ by $\mathbb{Z}$. Even as a monoid we need just two generators a and b and one relation $abba = 1$. There is no finite convergent system $S \subseteq \{a, b\}^* \times \{a, b\}^*$ such that $\{a, b\}^* / \{abba = 1\} = \{a, b\}^* / S$, but clearly such systems exists if we spend more generators. See [31] for more details about this example.

We finish the section with a few open problems.

Problem 2.6. *Is it true that every hyperbolic group has a finite convergent presentation?*

It is known that some hyperbolic groups have finite convergent presentations, for example, surface groups [34].

Problem 2.7. *Is it true that every finitely generated fully residually free group has a finite convergent presentation?*

The next two problems are from [43].

Problem 2.8. *Do all automatic groups have finite convergent presentations?*

Problem 2.9. *Do all one-relator groups have finite convergent presentations?*

Notice that all the groups above satisfy the homological condition FP_∞; which is the main known condition necessary for a group to have a finite convergent presentation, see [48, 47].

2.4 Computing with infinite systems

In this section we discuss computing with infinite systems. An infinite string rewriting system $S \subseteq \Gamma^* \times \Gamma^*$ can be used in computation if it satisfies some natural conditions. Firstly, one has to be able to recognise if a given pair $(u, v) \in \Gamma^* \times \Gamma^*$ gives a rule $u \to v \in S$ or not, i.e., the system S must be a recursive subset of $\Gamma^* \times \Gamma^*$. We call such systems *recursive*. Secondly, to rewrite with S one has to be able to check if for a given $u \in \Gamma^*$ there is a rule $\ell \to r \in S$ with $\ell = u$, so we assume that the set $L(S)$ of the left-hand sides of the rules in S is a recursive subset of Γ^*. Systems satisfying these two conditions are termed *effective* rewriting systems. Clearly, every finite system is effective. Notice also, that every recursive *non-length-increasing* system S (i.e., $|\ell| \geq |r|$ for every rule $\ell \to r \in S$) is effective. Indeed, given $u \in \Gamma^*$ one can check if a rule $u \to v$ is in S or not for all words v with $|v| \leq |u|$, thus effectively verifying whether $u \in L(S)$ or not.

The argument above shows that for a recursive non-length-increasing system S one can effectively enumerate all the rules in S in such a way

$$\ell_0 \to r_0, \ell_1 \to r_1, \ldots, \ell_i \to r_i, \ldots \tag{1}$$

that if $i < j$ then $\ell_i \preceq \ell_j$ in the length-lexicographical ordering $\preceq$ and also if $\ell_i = \ell_j$ then $r_i \preceq r_j$. We call this enumeration of S *standard*.

Proposition 2.10. *Let S be an infinite effective convergent system. Then the word problem in the monoid M_S defined by S is decidable.*

Proof. Given a word $u \in \Gamma^*$ one can start the rewriting process applying rules from S. Indeed, for a given factor w of u one can check if $w \in L(S)$ or not, thus enumerating all factors of u one can either find a factor w of u with $w \in L(S)$ or prove that u is S-irreducible. If such w exists one can enumerate all pairs (w, v) with $v \in \Gamma^*$ and check one by one if $(w, v) \in S$ or not. This procedure eventually terminates with a rule $w \to v \in S$. Now one can apply this rule to u and rewrite u into u_1. Applying again this procedure to u_1 one eventually arrives at a unique S-irreducible word $\widehat{u}$. To check if two words are equal in the monoid M_S one can find their S-irreducibles and check whether they are equal or not.

□

There are various modifications of the algorithm described above, that work for other types of, not necessarily convergent, infinite systems. We consider some of these below.

3 Length-reducing and Dehn systems

3.1 Finite length-reducing systems

In this section we study a very particular type of rewriting system, called *length-reducing* systems, where, for every rule $\ell \to r$ one has $|\ell| > |r|$. The main interest in length-reducing systems comes from the fact that, contrary to the case of finite convergent systems, the algorithm for computing the reduced forms is fast.

Lemma 3.1. *[7] If S is a finite length-reducing string rewriting system, then irreducible descendants of a given word can be computed in linear time (in the length of the word).*

This result is well-known, we use it in many parts of the paper, and it can be seen easily as follows.

Proof. First, we choose some $\varepsilon > 0$ such that $(1-\varepsilon)|\ell| \geq |r|$ for all rules $(\ell, r) \in S$.

Now, consider an input $w \in \Gamma^*$ of length $n = |w|$. For a moment, let a *configuration* be a pair (u, v) such that (i) $w \overset{*}{\underset{S}{\Longleftrightarrow}} uv$ and (ii) u is irreducible. The goal is to transform the initial configuration $(1, w)$ in $\mathcal{O}(n)$ steps into some final configuration $(\hat{w}, 1)$.

Say, we are in the configuration (u, v). The goal is achieved if $v = 1$. So assume that $v = av'$ where a is a letter. If ua is irreducible, then we replace (u, av') by (ua, v'), and (ua, v') is the next configuration. If however ua is reducible, then we can write $ua = u'\ell$ for some $(\ell, r) \in S$; and u' is irreducible. So, we replace (u, av') by (u', rv'), and (u', rv') is the next configuration. The algorithm is obviously correct. Defining the *weight* γ of configurations by $\gamma(u, v) = (1-\varepsilon)|u| + |v|$ we see that γ reduces from one configuration to the next by at least ε. Hence we have termination in linear time. □

In fact, length reducing rewriting systems arise naturally in the class of small cancellation groups, and more generally hyperbolic groups, which we might regard as a paradigm for groups with easily solvable word problem. To be precise: a group G is hyperbolic if and only if there is a finite generating set Γ for G and a finite length-reducing system $S \subseteq \Gamma \times \Gamma$ (so $G = \Gamma/S$) such that a word w represents the trivial element of G if and only if w can be S-reduced to the empty word, see [2]. In other words, a group is hyberbolic, if and only if there exists a finite length-reducing system which is confluent on the empty word.

Definition 3.2. *A length-reducing string rewriting system which is confluent on the empty word is called a* Dehn *system.*

If a group is defined by a finite length-reducing Dehn rewriting system then the rewriting algorithm is known in group theory as the *Dehn* algorithm. More general definitions of Dehn algorithms, to rewriting systems over a larger alphabet than the generators of the group, have been studied by Goodman and Shapiro [25] and Kambites and Otto [32]. In particular in [25] it is shown that such generalised Dehn algorithms solve the word problem in finitely generated nilpotent groups and many relatively hyperbolic groups.

It is known that, given a finite presentation of a hyperbolic group G, one can produce a finite Dehn presentation of G by adding, to a given presentation, all new relators of G up to some length (which depends on the hyperbolicity constant of G). However, this algorithm is very inefficient and the following questions remain.

Problem 3.3. *Is there a Knuth-Bendix type completion process that, given a finite presentation of a hyperbolic group G, finds a finite Dehn presentation of G.*

Problem 3.4. *Is there an algorithm that, given a finite presentation of a hyperbolic group, determines whether or not this presentation is Dehn.*

Notice that some partial answers to this question are known. Namely, in [3] Arzhantseva has shown that there is an algorithm that, given a finite presentation of a hyperbolic group and $\alpha \in [3/4, 1)$, detects whether or not this presentation is an α-Dehn presentation. Here a presentation $\langle X \mid R \rangle$ of a group G is called an α-Dehn presentation if any non-empty freely reduced word $w \in (X \cup X^{-1})^*$ representing the identity in G contains as a factor a word u which is also a factor of a cyclic shift of some $r \in R^{\pm 1}$ with $|u| > \alpha |r|$.

3.2 Infinite length-reducing systems

Let us discuss some algorithmic aspects of rewriting with infinite length-reducing systems.

Proposition 3.5. *Let $S \subseteq \Gamma^* \times \Gamma^*$ be an infinite recursive string rewriting system. Then the following hold.*

1) *If S is length-reducing then an irreducible descendant of a given word can be computed.*

2) *If S is Dehn and M_S is a group, then the word problem in M_S is decidable.*

Proof. The system S is effective since it is recursive and length-reducing (see remark before Proposition 2.10). Now the argument in the proof of Proposition 2.10 shows that for a given w one can effectively find an S-irreducible of w, so 1) and 2) follow. □

In the case of length-reducing systems one can try to estimate the time complexity of the algorithms involved. To this end we need the following definition. Let S be an effective non-length increasing rewriting system and

$$\ell_0 \to r_0, \ell_1 \to r_1, \ldots, \ell_i \to r_i, \ldots$$

its standard enumeration (see Section 2.4). If there an algorithm $\mathcal{A}$ and a polynomial $p(n)$ such that for every $n \in \mathbb{N}$ the algorithm $\mathcal{A}$ writes down the initial part of the standard enumeration of S with $|\ell_i| \leq n$ in time $p(n)$ then the system S is called *enumerable in time* $p(n)$ or *Ptime enumerable*. In particular, we say that S is linear (quadratic) time enumerable if the polynomial $p(n)$ is linear (quadratic).

Proposition 3.6. *Let $S \subseteq \Gamma^* \times \Gamma^*$ be an infinite non-length increasing string rewriting system, which is enumerable in time $p(n)$. Then the following hold.*

1) *If S is length-reducing then an irreducible descendant of a given word w can be computed in polynomial time.*

2) *If S is Dehn and M_S is a group, then the word problem in M_S is decidable in time in polynomial time.*

Proof. Given a word w one can list in time $p(|w|)$ all the rules $\ell \to r$ of the standard enumeration of S with $|\ell| \leq |w|$. Now in time $\mathcal{O}(p(|w|)|w|^2)$ one can check whether one of the listed rules can be applied to w or not. This proves 1) and 2). □

3.3 Weight-reducing systems

Many results above can be generalised to weight-reducing systems. A *weight* γ assigns to each generator a a positive integer $\gamma(a)$ with the obvious extensions to words by $\gamma(a_1 \cdots a_n) = \sum_{i=1}^{n} \gamma(a_i)$. A system is called *weight-reducing*, if for every rule $\ell \to r$ one has $\gamma(\ell) > \gamma(r)$. The following statements in this paragraph are taken from [16]. It is decidable whether a finite system is weight-reducing by linear integer programming. The reason to consider weight-reducing systems is that there are monoids like $\{ a, b, c \}^* / ab = c^2$ having an obvious finite convergent weight-reducing presentation, but where no finite convergent length-reducing presentation exists.

For groups the situation is unclear. Actually, the following conjecture has been stated.

Conjecture 3.7. *Let G be a finitely generated group. Then the following assertions are equivalent:*

1) *G is a* plain *group, i.e., G is a free product of free and finite groups.*

2) G has a finite convergent length-reducing presentation.

3) G has a finite convergent weight-reducing presentation.

The implications $1) \implies 2) \implies 3)$ are trivial, and $2) \implies 1)$ is known as the *Gilman conjecture* and was stated first in [23].

It is clear that the conjugacy problem can be decided in plain groups and this holds for groups G having a finite convergent weight-reducing presentation, too. In fact, for $s, t \in G$ the set $R_{s,t} = \{ g \in G \mid gsg^{-1} = t \}$ is an effectively computable rational subset of G.

4 Preperfect systems

4.1 General results

In this section we discuss preperfect rewriting systems, which play an important part in solving the word problem and finding geodesics (shortest representatives in the equivalence classes) in groups.

Definition 4.1. *A* Thue system *is a rewriting system $S \subseteq \Gamma^* \times \Gamma^*$ such that the following conditions hold:*

i) If $\ell \longrightarrow r \in S$ then $|\ell| \geq |r|$.

ii) If $\ell \longrightarrow r \in S$ with $|\ell| = |r|$, then $r \longrightarrow \ell \in S$, too.

To every rewriting system is associated an equivalent Thue system. In order to specify a Thue system which is equivalent to a rewriting system S one can do the following: symmetrize S by adding all the rules $r \longrightarrow \ell$ whenever $\ell \longrightarrow r \in S$, then throw out all the length increasing rules. The new system, denoted $T(S)$ is called the *Thue resolution* of S. It follows that every monoid has a Thue presentation.

Definition 4.2. *A confluent Thue system is called* preperfect.

The main interest in preperfect systems in algebra comes from the following known (and easy) complexity result: for which we require the following definition.

Definition 4.3. *A word $w \in \Gamma^*$ is termed S-*geodesic*, with respect to a string rewriting system S, if it has minimal length in its $\overset{*}{\underset{S}{\Longleftrightarrow}}$-equivalence class (and simply* geodesic *where no ambiguity arises).*

Clearly, S-geodesic words are precisely the geodesic words in the monoid Γ/S relative to the generating set Γ, i.e., they have minimal length among all the words in Γ^* that represent the same element in Γ/S. Sometimes, we say that a word $w \in \Gamma^*$ is a geodesic of a word $u \in \Gamma^*$, if w is S-geodesic and $\overset{*}{\underset{S}{\Longleftrightarrow}}$-equivalent to u.

Proposition 4.4. *If a rewriting system S is finite and preperfect, then one can decide the word problem in the monoid defined by S in polynomial space, and hence in exponential time. Moreover, along the way one can find an S-geodesic of a given word w, as well as, all S-geodesics of w.*

A locally confluent (strictly) length-reducing system is convergent, hence, from the above, preperfect. However the Thue resolution of an arbitrary finite convergent rewriting system may fail to be terminating or confluent as simple examples show. (Let $\Gamma = \{a, b, c, d, u, v\}$ and S be the system with rules $ab \longrightarrow u$, $bc \longrightarrow v$, $uc \longrightarrow d^3$ and $av \longrightarrow d^3$. Then $T(S)$ is not confluent. The system with one rule $a \longrightarrow b$ has non-terminating Thue resolution.) It is also easy to see that $T(S)$ may be preperfect when S is not confluent. On the other hand, if a confluent system S has no length-increasing rules, then the Thue resolution can be constructed by symmetrizing S relative to all length preserving rules in S (by adding the rule $r \longrightarrow \ell$ for each length preserving rule $\ell \longrightarrow r \in S$) and a straightforward argument shows that in this case $T(S)$ is confluent, so preperfect.

Lemma 4.5. *If S is a confluent rewriting system with no length-increasing rules then the Thue resolution $T(S)$ is preperfect.*

For a system S (for example a Thue system) where all rules $\ell \to r \in S$ are either length-reducing $|\ell| > |r|$, or length-preserving $|\ell| = |r|$, it is convenient to split S into a length reducing part S_R and a length preserving part S_P, so $S = S_R \cup S_P$. If S is a Thue system then all S-geodesic words that lie in the same equivalence class have the same length and any two of them are S_P-equivalent (can be transformed one into other by a sequence of rules from S_P). Therefore, the word length in Γ^* induces a well-defined length on the factor-monoid $M = \Gamma^*/S_P$ (application of relations from S_P does not change the length). Hence, one can view S_R as a length reducing rewriting system over the monoid $M = \Gamma^*/S_P$, in which case we assume that $S_R \subseteq M \times M$. Note that if S_P is finite and if there is an effective way to perform reduction steps with S_R, then the word problem in M is decidable.

Decidability of the word problem in $M = \Gamma^*/S_P$ allows one to test whether a given rule from S_R is applicable to an element of M. Since $S_R \subseteq M \times M$ is terminating it suffices to show local confluence to ensure convergence. This may tempt one to introduce an analogue of the Knuth-Bendix completion. However, in general an infinite number of *critical pairs* may appear in the Knuth-Bendix process, and one needs to be able to recognise when the current system becomes preperfect. Unfortunately, this is algorithmically undecidable. More precisely, the following result holds.

Theorem 4.6 ([38]). *The problem of verifying whether a finite Thue system is preperfect or not is undecidable.*

In fact in [39] this problem is shown to be undecidable even in the case of a Thue system, whose length-preserving part S_P consists only of a single rewriting rule of the form $ab \longleftrightarrow ba$. On the other hand, under some additional assumptions such a procedure can yield useful results ([17, 18]) - good examples in our context are graph groups, c.f. Section 7.1.

In the final part of this section we discuss some complexity issues in computing with preperfect systems. By Proposition 4.4 finite preperfect systems allow one to solve the word problem and find geodesics in at most exponential time.

Proposition 4.7. *Let S be an infinite preperfect rewriting system. Then:*

1) *if S is recursive then the word problem in the monoid M_S defined by S is decidable;*

2) *if S is Ptime enumerable then one can solve the word problem in M_S and find a geodesic of a given word in exponential time.*

Proof. Since preperfect rewriting systems are non-length-increasing it follows that recursive preperfect systems are effective. (see the remark before Proposition 2.10). Therefore, given a word w one can effectively list all the rules in S with the left-hand sides of length at most $|w|$. Denote this subsystem of S by S_w. Now rewriting w using S is exactly the same as using S_w, so 1) and 2) follow from the argument in the proof of Proposition 4.4 for finite preperfect systems.

□

5 Geodesically perfect rewriting systems

In this section we consider a subclass of Thue systems which are designed to deal with geodesics in groups or monoids. In particular, we study confluent geodesic systems, which form a subclass of preperfect string rewriting systems, and which behave better in many ways than general preperfect systems. We call these systems *geodesically perfect*, as this indicates their essential properties and fits with the terminology of preperfect systems. However as discussed in Section 1 they are also known in the literature as almost confluent or quasi-perfect. The motivation for the study of geodesically perfect systems in group theory comes mainly from attempts to solve the, algorithmically difficult, *geodesics problem*: that is, given a finite presentation of a group G and a word w in the generators, find a word of minimal length representing w as an element of G.

5.1 Geodesic systems

We consider first a somewhat larger, less well-behaved, class of rewriting systems.

Definition 5.1. *A string rewriting system $S \subseteq \Gamma^* \times \Gamma^*$ is called* geodesic *if S-geodesic words are exactly those words to which no length reducing rule from S can be applied.*

Note that if S is a geodesic rewriting system then its Thue resolution is also geodesic, this allows us to assume, without loss of generality, that geodesic systems are Thue systems.

Remark 5.2. *Dehn rewriting systems are not in general geodesic: they need only rewrite words that represent the identity to (empty) geodesics in Γ^*/S.*

A finite geodesic system gives a linear time algorithm to find a geodesic of a given word $u \in \Gamma^*$. The following algebraic characterisation of finite geodesic systems in groups is given in [24] (the definition of geodesic in [24] is slightly more restrictive than ours, however this makes no difference to the result).

Theorem 5.3 ([24])**.** *A group G is defined by a finite geodesic system S if and only if G is a finitely generated virtually free group.*

From the result of Rimlinger quoted above finitely generated virtually free groups are precisely the universal groups of finite pregroups. It follows that every finite length reducing geodesic system can be transformed to the length reducing part of the rewriting system (see Section 8.1) associated with a finite pregroup.

The following result follows from Proposition 3.6.

Proposition 5.4. *Let S be a geodesic Ptime enumerable string rewriting system such that the monoid M_S is a group. Then the word problem in the group M_S is decidable in polynomial time.*

Very little is known about geodesic systems which do not present groups. In particular, it is not clear whether the word problem remains decidable: that is, given $u, v \in \Gamma^*$ decide whether or not $u \overset{*}{\underset{S}{\Longleftrightarrow}} v$.

Problem 5.5. *Does there exist a finite geodesic system S for which the word problem is undecidable?*

The following result demonstrates one of the principal difficulties of working with geodesic systems.

Theorem 5.6. *It is undecidable whether a finite rewriting system is geodesic.*

Proof. The proof is a modification of the proof by Narendran and Otto [39] which showed undecidability of preperfectness in presence of a single commutation rule.

We need some notation and we adhere as far as possible to that of [39]. We shall define the computation of a Turing machine by a set of rewriting rules. A configuration of the machine is then a particular form of word over the tape alphabet, the states and the end markers. In detail let Σ be a finite set, the tape alphabet, let $\overline{\Sigma}$ be a disjoint copy of Σ, let Q be a finite set of states, and α and β be special symbols representing end markers. There are two marked states q_0 and q_f, the initial and final states. The computation of the machine can be described by a finite set of rules which fall into the following categories, where we use the notation $p, q \in Q$, $p \neq q_f$, $a, a', b \in \Sigma$:

1.) $pa \longrightarrow \overline{a'}q$.

   (Read a in state p, write a', move one step to the right, switch to state q.)

2.) $\overline{b}pa \longrightarrow qba'$. (As above, but move one step to the left.)

3.) $p\beta \longrightarrow pa\beta$. (Create new space before the right end marker.)

These rewriting rules constitute the rewriting system associated to M. We assume that the machine is deterministic, so there no overlapping rules. A *configuration* of a (deterministic) Turing machine is then a word $\alpha\overline{u}qv\beta$ with $\overline{u} \in \overline{\Sigma}^*$, $v \in \Sigma^+$, and $q \in Q$. The initial configuration on input $x \in \Sigma^*$ is the word $\alpha q_0 x\beta$. We assume that the machine stops if and only if it reaches the state q_f.

Now let M be a Turing machine for which it is undecidable whether or not computation halts on on input $x \in \Sigma^*$. Using this machine we are going to construct, for each $x \in \Sigma^*$, a new length reducing rewriting system S_x, which is geodesic if and only if the machine M does not stop on input x.

The alphabet Γ of each such system is to consist of the symbols of $\Sigma \cup \overline{\Sigma} \cup Q \cup \{\alpha, \beta\}$ and new additional symbols $d, e, \gamma, \delta, I, C$. The system S_x will consist of rules which simulate the computation of M on input x, with some additional control on the number of steps of the computation carried out. Let $x \in \Sigma^*$. To begin with, we introduce rules leading to two different initial configurations. Let $m = |x| + 5$. We define the two rules

$$\alpha q_0 x\beta\gamma \underset{(1)}{\longleftarrow} IC^m \underset{(2)}{\longrightarrow} \alpha q_0 x\beta\delta.$$

Next we introduce rules, involving d, e, γ and δ, to control the number of steps of the simulation. Symbols γ and δ convert d's to e's. The latter act as tokens to control the number of steps performed by the simulation of M. Both γ and δ move right consuming three d's and producing two e's, the difference being that between γ may move to the right arbitrarily far from β whereas δ is forced to remain very close to β. The effect on the length of a word of each rule is the same.

Explicitly, we add new rules of the form:

$$\gamma ddd \longrightarrow ee\gamma, \quad \beta\delta ddd \longrightarrow \beta ee\delta.$$

Note that, using rule (1), all words in $IC^m(ddd)^*$ now reduce as follows

$$IC^m d^{3n} \underset{(1)}{\Longrightarrow} \alpha q_0 x\beta\gamma d^{3n} \overset{*}{\Longrightarrow} \alpha q_0 x\beta e^{2n}\gamma.$$

However, using the rule $IC^m \underset{(2)}{\longrightarrow} \alpha q_0 x\beta\delta$ in the first step we can only do

$$IC^m d^{3n} \underset{(2)}{\Longrightarrow} \alpha q_0 x\beta\delta d^{3n} \overset{*}{\Longrightarrow} \alpha q_0 x\beta ee\delta d^{3n-3}$$

and then, for $n \geq 2$, we are stuck.

Now we bring e into the game. The letter e is used to *enable* a computation step of M. It can move to the left until it is at distance one to the right of a state symbol. The *generic* rules for e allow e to move left and are as follows:

$$abee \longrightarrow aeb, \quad aebee \longrightarrow aeeb \quad \text{for } a \in \Sigma, b \in \Sigma \cup \{\beta\}.$$

Let us describe the effect of these rules on words of the form

$$\alpha\overline{u}pa_1 \cdots a_k\beta\delta d^{3n}$$

where n is huge (and k is viewed as constant $k \geq 0$), $a_i \in \Sigma$, $\overline{u} \in \overline{\Sigma}^*$. The maximal possible reduction leads to a word of the form

$$\alpha\overline{u}pa_1 ee \cdots a_k ee\beta ee\delta d^{3n'}.$$

In this case, if n is large enough, then $n' > 0$, no further reduction is possible and actually $n - n' \in \mathcal{O}(1)$.

At this point we introduce rules to simulate the computation of the machine M. There is one simulation rule corresponding to each rule in the rewriting system associated to M. More precisely we introduce a rule $uee \longrightarrow v$ for each rewriting

I.) Initial rules:

$$\alpha q_0 x\beta\gamma \underset{(1)}{\longleftarrow} IC^m \underset{(2)}{\longrightarrow} \alpha q_0 x\beta\delta.$$

II.) Step control rules, for $a \in \Sigma$, $b \in \Sigma \cup \{\beta\}$:

$$\gamma ddd \longrightarrow ee\gamma,$$

$$\beta\delta ddd \longrightarrow \beta ee\delta,$$

$$abee \longrightarrow aeb,$$

$$aebee \longrightarrow aeeb.$$

III.) Simulation rules, for $a, b \in \Sigma$, $p \in Q\backslash\{q_f\}$:

$$paee \longrightarrow \overline{a'}q,$$

$$\overline{b}paee \longrightarrow qba',$$

$$p\beta ee \longrightarrow pa\beta.$$

Figure 1: The system S_x.

rule $u \longrightarrow v$ of M: so we have simulation rules of three types (where again we use the notation $p, q \in Q$, $a, a', b \in \Sigma$)

$$paee \longrightarrow \overline{a'}q, \quad \overline{b}paee \longrightarrow qba', \quad p\beta ee \longrightarrow pa\beta.$$

The system S_x consists of the rules defined so far, which we list in Figure 1, so is length reducing.

Now assume that the machine M halts on input x. This implies that only finitely many computation steps t can be performed. Again choose n huge and view t and $|x|$ as constants. Consider a word of the form $IC^m d^{3n}$. Starting a reduction with the second rule we get stuck at an irreducible word when the simulation reaches state q_f:

$$IC^m d^{3n} \underset{(2)}{\longrightarrow} \alpha q_0 x\beta\delta d^{3n} \overset{*}{\Longrightarrow} \alpha\overline{u}q_f a_1 ee\cdots a_k ee\beta ee\delta d^{3n'}$$

at which point $n - n' \in \mathcal{O}(1)$. The system S_x cannot be geodesic because with the other initial rule we can first move Γ to the right of all the d's thereby losing n letters immediately:

$$IC^m d^{3n} \underset{(1)}{\longrightarrow} \alpha q_0 x\beta\gamma d^{3n} \overset{*}{\Longrightarrow} \alpha q_0 x\beta d^{2n}\gamma$$

IV.) Completion rules:

$$\forall i \geq 0: \ \alpha w_i \beta \delta \longrightarrow \alpha w_i \beta \gamma$$

where $\alpha w_i \beta ee \underset{S_x}{\overset{*}{\Longrightarrow}} \alpha w_{i+1}\beta, \forall i \geq 0,$ and $w_0 = q_0 x.$

Figure 2: The additional rules of system T_x.

and then when the simulation reaches the state q_f the resulting irreducible word will end $e^{2n'}\gamma$ instead of $\delta d^{3n'}$ (and otherwise will be the same).

It remains to cover the case when the machine does not halt on input x. We shall show that in this case the system S_x is geodesic. Note that, as M never reaches state q_f, for all $n > 0$

$$\alpha q_0 x \beta e^{2n} \underset{S_x}{\overset{*}{\Longrightarrow}} \alpha \overline{u} p y \beta,$$

where $\overline{u} \in \overline{\Sigma}^*$, $p \in Q$ and $y \in (\Sigma \cup \{e\})^*$. For technical reasons we define a sequence of words w_i for $i \geq 0$ as follows. We let $w_0 = q_0 x$ and let $\alpha w_{i+1}\beta$ be defined to be the irreducible descendant of $\alpha w_i \beta ee$. The sequence of words w_i is infinite because the machine does not stop on input x. Thus

$$\forall i \geq 0: \ \alpha w_i \beta ee \underset{S_x}{\overset{*}{\Longrightarrow}} \alpha w_{i+1}\beta \in \mathrm{IRR}(S_x).$$

Now we add infinitely many rules to S_x to form a new system T_x as follows:

$$\forall i \geq 0: \ \alpha w_i \beta \delta \longrightarrow \alpha w_i \beta \gamma$$

As the rules of T_x are generated by steps of the Knuth-Bendix completion procedure applied to S_x the congruences generated by S_x and T_x are the same. To summarise, the system T_x consists of the rules of Figure 1 and those listed in Figure 2. Thus T_x is is terminating and local confluence can be checked directly.

Each word $w \in \Gamma^*$ has a unique factorisation where we choose k and all n_j to be maximal:

$$u_0(IC^m d^{n_1})u_1 \cdots (IC^m d^{n_k})u_k.$$

The benefit of the system T_x is that it provides us with canonical geodesics. A geodesic of w is given by:

$$\widehat{u_0}(\alpha w_{i_1}\beta\gamma d^{m_1})\widehat{u_1} \cdots (\alpha w_{i_k}\beta\gamma d^{m_k})\widehat{u_k},$$

where $m_j = n_j \bmod 3$. The crucial observation is that allowing only rules from S_x we achieve exactly the same form with the exception that some γ's are still δ's. Thus, the system is geodesic. □

5.2 Geodesically perfect systems

Definition 5.7. *A string rewriting system $S \subseteq \Gamma^* \times \Gamma^*$ is called* geodesically perfect, *if*

i) S is geodesic, and

ii) if $u, v \in \Gamma^*$ are S-geodesics, then $u \overset{*}{\underset{S}{\Longleftrightarrow}} v$ if and only if $u \overset{*}{\underset{S_P}{\Longleftrightarrow}} v$, where S_P is the length-preserving part of S.

Again, it follows directly that if S is a geodesically perfect system then so is its Thue resolution, so we can assume that geodesically perfect systems are Thue. If S is a geodesically perfect Thue system then we write it as $S = S_R \cup S_P$ where S_R is its length reducing part and S_P its length preserving part. It also follows from the definition that a geodesically perfect system is confluent.

There is a simple procedure to describe geodesics of elements in the monoid Γ^*/S defined by a geodesically perfect Thue system S. Namely, the geodesics of a given word $w \in \Gamma^*$ are the S_R-reduced forms of w and any two such geodesics can be obtained from one another by applying finitely many rules from S_P. Moreover it is shown in [6] that the word problem for monoids defined by finite geodesically perfect rewriting systems is PSPACE complete.

The following result relates geodesically perfect to preperfect Thue systems.

Proposition 5.8. *Let $S \subseteq \Gamma^* \times \Gamma^*$ be a Thue system. Then*

1) if S is geodesically perfect then it is preperfect and

2) if S is preperfect and geodesic then it is geodesically perfect.

Proof. 1) follows from the observation that geodesically perfect implies confluent. To see 2) observe that S is confluent, hence Church-Rosser. Therefore, if u, v are two geodesics with $u \overset{*}{\underset{S}{\Longleftrightarrow}} v$ then $u \overset{*}{\underset{S}{\Longrightarrow}} w$ and $w \overset{*}{\underset{S}{\Longleftarrow}} v$ for some $w \in \Gamma^*$. Since u, v are S-geodesics the only rules that could be applied in $u \overset{*}{\underset{S}{\Longrightarrow}} w$ and $w \overset{*}{\underset{S}{\Longleftarrow}} v$ are length preserving, hence $u \overset{*}{\underset{S_P}{\Longleftrightarrow}} v$, as required. □

In Section 8.1 we will describe a general tool to construct geodesically perfect systems defining groups: based on the fact that rewriting systems associated with pregroups are always geodesically perfect.

In Corollary 8.7 we prove that groups defined by finite geodesic systems are exactly the groups defined by finite geodesically perfect systems.

Obviously, every geodesic rewriting system S contains the length-reducing part T_R of some (infinite) geodesically perfect Thue system T defining the same monoid. Indeed, one can obtain T by first constructing the Thue resolution T' of S and then adding length-preserving rules to T' to make it confluent. But it is not true that every finite geodesic rewriting system S is the length-reducing part of a finite geodesically perfect system defining the same monoid. To see this consider the following example.

Example 5.9. *The following system is geodesic, and it is not the length-reducing part of any finite geodesically perfect system defining the same quotient monoid.*

$$add \longrightarrow ab, \quad add \longrightarrow ac, \quad bdd \longrightarrow eb, \quad cdd \longrightarrow ec.$$

Indeed, let S be the system above, and let $T = S \cup \{ b \longleftrightarrow c \}$. The new system T is geodesically perfect by Proposition 6.1. But T-geodesics are computed by using rules from S. As $\overset{*}{\underset{S}{\Longleftrightarrow}} \subseteq \overset{*}{\underset{T}{\Longleftrightarrow}}$ we see that S is a geodesic system.

Let us show that S is not the length-reducing part of any equivalent, finite, geodesically perfect system. For a contradiction, assume that a finite set T of non-trivial symmetric rules can be added to S such that $S \cup T$ becomes geodesically perfectand is equivalent to S. Assume T involves a new letter, say f. Then f is equal to some word u_f over $\{a, b, c, d, e\}$ which is irreducible with respect to S. If u_f is empty, then we do not need f, hence u_f is nonempty and we have $u_f \overset{*}{\underset{T}{\Longrightarrow}} f$. The rules of T are symmetric (hence length preserving), so f is accompanied by a rule, say $f \longleftrightarrow a$, and f is redundant. So, actually we may assume $T \subseteq \{a, b, c, d, e\}^* \times \{a, b, c, d, e\}^*$. Clearly, $ae^nb \overset{*}{\underset{S}{\Longleftarrow}} ad^{2n+2} \overset{*}{\underset{S}{\Longrightarrow}} ae^nc$, hence $ae^nb \overset{*}{\underset{S}{\Longleftrightarrow}} ae^nc$ and ae^nb, ae^nc are in the same class and are S-reduced. Because T is finite, some left hand side of T must contain a word in $u \in ae^* \cup e^* \cup e^*b \cup e^*c$. But all these words u are S-reduced, hence geodesic. Moreover, for any such u there is no other word v in the same class as u and of the same length. So, for large enough n the rules of T cannot be applied to either ae^nb or ae^nc. As T is the length preserving part of the supposedly geodesically perfect system $S \cup T$, this is the required contradiction.

Remark 5.10. *Let $M = \{a, b, c, d, e\}^*/S$ be the quotient monoid as in Example 5.9. The proof above can be modified in order to show that actually there is no finite system $T \subseteq \{a, b, c, d, e\}^* \times \{a, b, c, d, e\}^*$ which is geodesically perfect and which defines M. However, if we use an additional letter f, then the following system defines M, too.*

$$dd \longrightarrow f, \quad af \longleftrightarrow ab, \quad af \longleftrightarrow ac, \quad bf \longleftrightarrow eb, \quad cf \longleftrightarrow ec.$$

The system is geodesically perfect, by Proposition 6.1 again.

We note that Example 5.9 illustrates a general fact: namely that if S is a rewriting system and there exists a set T of symmetric rules such that $S \cup T$ is geodesically perfect (but not necessarily equivalent to S) then S itself is geodesic. Since geodesic systems are undecidable whereas geodesically perfect systems are decidable this could prove to be a useful test for a geodesic system.

6 Knuth-Bendix completion for geodesically perfect systems

A classical result of Nivat and Benois (stated in Proposition 6.1) shows that it is decidable whether a finite Thue system is geodesically perfect. In order to explain the criterion we need the notion of *critical pair*. All rewriting systems S in this subsection are viewed as Thue systems and split into a length reducing part S_R and a length preserving part of symmetric rules S_P. By definition, a critical pair is a pair (x, y) arising from the situation

$$x \underset{(\ell_1,r_1)}{\Longleftarrow} z \underset{(\ell_2,r_2)}{\Longrightarrow} y$$

subject to the following conditions:

1. $(\ell_1, r_1) \in S_R$ is length reducing, but $(\ell_2, r_2) \in S$ can be any rule.
2. $z = \ell_i u_i = u_j \ell_j$ with $|u_i| < |\ell_j|$ and $i, j \in \{1, 2\}$ such that $i = j$ implies $u_i = u_j = 1$.

Proposition 6.1 ([41]). *A finite Thue system S is geodesically perfect if and only if for all critical pairs (x, y) there are words x' and y' such that with length reducing reductions we have:*

$$x' \overset{*}{\underset{S_R}{\Longleftarrow}} x, \quad y \overset{*}{\underset{S_R}{\Longrightarrow}} y',$$

and with length preserving reductions we have:

$$x' \overset{*}{\underset{S_P}{\Longleftrightarrow}} y'$$

Proof. The proof is not very difficult and can be found, for example, in the book [6, Thm. 3.6.4]. □

Remark 6.2. *Note that the words x' and y' in Proposition 6.1 need not be irreducible w.r.t. the length reducing subsystem S_R. This fact is actually used in the proof of Proposition 6.3.*

This criterion leads to the following version of the Knuth-Bendix procedure. Consider a finite Thue system S_0. We shall construct a series of Thue systems $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots$ such that the union over all S_i is geodesically perfect and we have $S_i = S_{i+1}$ (i.e., the completion procedure stops) if and only if there exists a finite Thue system T which is geodesically perfect and equivalent to S_0, that is $\overset{*}{\underset{S_0}{\Longleftrightarrow}} = \overset{*}{\underset{T}{\Longleftrightarrow}}$. We divide the procedure into phases. We assume that in phase i a Thue system $S_i = S_R \cup S_P$ has been defined such that S_R contains the length reducing rules, S_P contains the length preserving rules, and $\overset{*}{\underset{S_0}{\Longleftrightarrow}} = \overset{*}{\underset{S_i}{\Longleftrightarrow}}$.

We begin phase $i+1$ by computing a list of all critical pairs of the system S_i (which were not already considered in phases 1 to i). For each such pair (x, y) choose words $\widehat{x}, \widehat{y}$, irreducible with respect to the subsystem S_R, such that

$$\widehat{x} \overset{*}{\underset{S_R}{\Longleftarrow}} x, \quad y \overset{*}{\underset{S_R}{\Longrightarrow}} \widehat{y}.$$

Define new rules as follows.

- If $|\widehat{x}| > |\widehat{y}|$ then add the rule $\widehat{x} \longrightarrow \widehat{y}$ to S_R.
- If $|\widehat{y}| > |\widehat{x}|$ then add the rule $\widehat{y} \longrightarrow \widehat{x}$ to S_R.
- If $|\widehat{y}| = |\widehat{x}|$ then test whether or not

$$\widehat{x} \overset{*}{\underset{S_P}{\Longleftrightarrow}} \widehat{y}.$$

  If the answer is negative then add the symmetric rule $\widehat{x} \longleftrightarrow \widehat{y}$ to S_P.

The system S_{i+1} is defined to be S_i together with all new rules which have been added to resolve all critical pairs of S_i. On a formal level we define S_i for all $i \geq 0$, but, of course, the procedure stops as soon as $S_i = S_{i+1}$, i.e., no new rules are needed to resolve critical pairs of S_i. Thus, if it stops with $S_i = S_{i+1}$ then S_i is a finite geodesically perfect Thue system, which is equivalent to S_0 (and we have $S_i = S_j$ for all $i \leq j$). However, what we really wish is stated in the following proposition.

Proposition 6.3. *Let $S_0 = S_R \cup S_P$ be a finite Thue system with length reducing rules S_R and length preserving rules S_P. Let*

$$S_0 \subseteq S_1 \subseteq \cdots S_i \subseteq \cdots$$

be the sequence of Thue systems which are computed by the Knuth-Bendix completion as described above. Let $\widetilde{S} = \bigcup_{i \geq 0} S_i$. Then the system $\widetilde{S}$ is geodesically perfect and we have $\overset{*}{\underset{S_0}{\Longleftrightarrow}} = \overset{*}{\underset{S_i}{\Longleftrightarrow}} = \overset{*}{\underset{\widetilde{S}}{\Longleftrightarrow}}$ for all $i \geq 0$. Moreover the following statements are equivalent.

1.) We have $S_i = S_{i+1}$ for some $i \geq 0$.

2.) The Thue system $\widetilde{S}$ is finite and geodesically perfect.

3.) There exists some finite geodesically perfect Thue system T such that $\underset{S_0}{\overset{*}{\Longleftrightarrow}} = \underset{T}{\overset{*}{\Longleftrightarrow}}$.

Proof. If $S_i = S_{i+1}$ for some $i \geq 0$, then clearly $\widetilde{S}$ is finite. It is geodesically perfect by the criterion of Nivat and Benois, c.f. Proposition 6.1. So, assume there exists some finite geodesically perfect Thue system T with $\underset{S_0}{\overset{*}{\Longleftrightarrow}} = \underset{T}{\overset{*}{\Longleftrightarrow}}$. We have to show that the procedure stops. We let m be large enough that $m \geq \max\{\, |\ell| \mid (\ell, r) \in T \,\}$. Next we consider i large enough that S_i contains all rules from $\widetilde{S}$ where the left hand side has length of at most m. Clearly, an index $i \in \mathbb{N}$ with this property exists. We will show that S_i is geodesically perfect, and Proposition 6.1 immediately implies $S_i = S_{i+1}$. For technical reasons, in a first step we we remove from T all length preserving rules $(\ell, r) \in T$ where we can apply to ℓ a length reducing rule of T. It is clear that new and smaller system T' is still geodesically perfect and $\underset{S_0}{\overset{*}{\Longleftrightarrow}} = \underset{T'}{\overset{*}{\Longleftrightarrow}}$; so we replace T with T'. Since now $(\ell, r) \in T$ with $|\ell| = |r|$ implies that ℓ and r are geodesics and since $\widetilde{S}$ is geodesically perfectand m is large enough, we see that $\underset{T_P}{\overset{*}{\Longleftrightarrow}} \subseteq \underset{(S_i)_P}{\overset{*}{\Longleftrightarrow}}$.

Next consider some word $\widehat{x}$, which is irreducible with respect to the length reducing rules in S_i. The claim is that $\widehat{x}$ is a geodesic. Indeed assume the contrary. Then a length reducing rule $(\ell, r) \in T$ can be applied to $\widehat{x}$. Since ℓ is not geodesic, there is a length reducing rule in $\widetilde{S}$ which can be applied to ℓ, but due to the definition of m this rule is in S_i, too. Thus, we have a contradiction and so S_i is geodesic. Now suppose that $\widehat{x}$ and $\widehat{y}$ are geodesic and that $\widehat{x} \underset{S_i}{\overset{*}{\Longleftrightarrow}} \widehat{y}$. Then $\widehat{x} \underset{T}{\overset{*}{\Longleftrightarrow}} \widehat{y}$ so $\widehat{x} \underset{T_P}{\overset{*}{\Longleftrightarrow}} \widehat{y}$. As $\underset{T_P}{\overset{*}{\Longleftrightarrow}} \subseteq \underset{(S_i)_P}{\overset{*}{\Longleftrightarrow}}$ this implies $\widehat{x} \underset{(S_i)_P}{\overset{*}{\Longleftrightarrow}} \widehat{y}$ so S_i is geodesically perfect. □

A finite geodesic (or geodesically perfect) rewriting system $S \subseteq \Gamma^* \times \Gamma^*$ allows one to find S-geodesics in linear time. In particular, if the monoid $M = \Gamma^*/S$ defined by S is a group one can solve the word problem in M in linear time. However, in general, there seems to be no linear time reduction from the word problem in a monoid M to the geodesic problem.

7 Examples of preperfect systems in groups

7.1 Graph groups

Let $\Delta = (\Sigma, E)$ be an undirected graph. The *graph group* (or a *right angled Artin group*, or a *partially commutative group*) defined by Δ is the group $G(\Delta)$ given by the presentation

$$G(\Delta) = F(\Sigma)/\{\, ab = ba \mid (a,b) \in E \,\},$$

where $F(\Sigma)$ is the free group with basis Σ. The group $G(\Delta)$ has a monoid presentation given by a preperfect rewriting system S_Δ. Indeed, let $\Gamma = \Sigma \cup \overline{\Sigma}$ where $\overline{\Sigma}$ is a disjoint copy of Σ. The rules of S_Δ are:

$$\begin{array}{lcll} a\overline{a} & \longrightarrow & 1 & \\ ab & \longleftrightarrow & ba & \text{if } \left\{ (a,b), (\overline{a},b), (a,\overline{b}), (\overline{a},\overline{b}) \right\} \cap E \neq \emptyset \end{array}$$

where $a, b \in \Gamma$ and $\overline{\overline{a}} = a$ for all $a \in \Gamma$.

If the graph Δ is finite the system S_Δ provides us with a decision algorithm for solving the word problem in $G(\Delta)$, though not the fastest one (WP in graph groups can be solved in linear time, see [53, 18]). However, the system S_Δ is very intuitive and simple, and it gives the geodesics in $G(\Delta)$, which are precisely the words whose length cannot be reduced by S_Δ.

Although it is preperfect the system S_Δ is not geodesically perfect. However every graph group may be constructed by a sequence of HNN-extensions and free products with amalgamation, starting with infinite cyclic groups, and so, from the results of Section 8 below it follows that these groups may be defined by (infinite) geodesically perfect systems. Moreover finite convergent rewriting systems for these groups have been found by Hermiller and Meier [27] (see also [5, 22, 52]).

7.2 Coxeter groups

Let $D_3 = \{a,b\}^*/\{a^2 = 1, b^2 = 1, (ab)^3 = 1\}$ be a dihedral group. Define a preperfect system S by the following rules

$$\begin{array}{lcl} aa & \longrightarrow & 1, \\ bb & \longrightarrow & 1, \\ aba & \longleftrightarrow & bab. \end{array}$$

More generally, a *Coxeter group* on n generators $a_1, \ldots, a_n$ is given by a symmetric $n \times n$ matrix (m_{ij}) with entries in $\mathbb{N}$ and 1's on the diagonal. The defining relations are given by:

$$(a_i a_j)^{m_{ij}} = 1 \quad \text{for all } 1 \le i, j \le n.$$

Note that this implies $a_i^2 = 1$ since $m_{ii} = 1$; and if $m_{ij} = 0$, then the equation $(a_i a_j)^0 = 1$ is trivial. (Therefore it is also common to write $(a_i a_j)^\infty = 1$, because $a_i a_j$ turns out to be an element of infinite order in this case.)

The word problem of Coxeter groups can be solved by the preperfect *Tits system* [51] (see also ([9, 1, 12]) of rewriting rules:

$$\begin{array}{rll} a_i^2 & \longrightarrow 1, & \text{for } 1 \le i \le n, \\ (a_i a_j a_i a_j \cdots) & \longleftrightarrow (a_j a_i a_j a_i \cdots) & \text{for } 1 \le i, j \le n \text{ and} \\ & & |(a_i a_j a_i a_j \cdots)| = |(a_j a_i a_j a_i \cdots)| = m_{ij}. \end{array}$$

The classical proof that this system is preperfect relies on the fact that Coxeter groups are linear [4]. Of course this system is not geodesically perfect. For virtually free Coxeter groups Corollary 8.7 guarantees the existence of a finite geodesically perfect rewriting system. It is shown in [26] that every Coxeter group is either virtually free or contains a surface group; but the question of whether the latter can be defined by a geodesically perfect system (necessarily infinite) remains open.

Convergent rewriting systems for Coxeter groups have been constructed, using the Knuth-Bendix procedure, by le Chenadec [34], but in general these are not finite. Finite convergent rewriting systems for certain classes of Coxeter groups have been found by Hermiller [28] (see also [19, 8]).

7.3 HNN-extensions

Let G be any group with isomorphic subgroups A and B. Let $\Phi : A \to B$ an isomorphism and let t be a fresh letter. By $\langle G, t \rangle$ we mean the free product of G with the free group $F(t)$ over t. The HNN-extension of G by (A, B, Φ) is the quotient group

$$\mathrm{HNN}(G; A, B, \Phi) = \langle G, t \rangle \,/\, \left\{\, t^{-1} a t = \Phi(a) \;\middle|\; a \in A \,\right\}$$

There is normal form theorem for elements in $\mathrm{HNN}(G; A, B, \Phi)$, which implies that G embeds into $\mathrm{HNN}(G; A, B, \Phi)$ and shows under which restrictions decidability of the word problem for G transfers to HNN-extensions. Usually the normal form theorem is shown by appeal to a combination of arguments of Higman, Neumann and Neumann and Britton, see [35, Chapter IV, Theorem 2.1].

Another option is to define a convergent string rewriting system. To see this, let $\Gamma = \{\, t, t^{-1} \,\} \cup G \setminus \{1\}$ and view Γ as a possibly infinite alphabet. We identify

$1 \in G$ with the empty word $1 \in \Gamma^*$. We choose transversals for cosets of A and B. This means we choose $X, Y \subseteq G$ such that there are unique decompositions

$$G = AX = BY$$

We may assume that $1 \in X \cap Y$.

The system $S \subseteq \Gamma^* \times \Gamma^*$ is now defined by the following rules with the convention that $[gh]$ denotes $gh \in G$ (as a single letter or the empty word).

$$\begin{array}{rcll} t^{-1}t & \longrightarrow & 1; & tt^{-1} \longrightarrow 1; \quad gh \longrightarrow [gh], \text{ for all } g, h \in G; \\ tg & \longrightarrow & aty, & \text{if } a \in A,\ a \neq 1,\ y \in Y,\ \Phi(a)y = g \text{ in } G; \\ t^{-1}g & \longrightarrow & bt^{-1}x, & \text{if } b \in B,\ b \neq 1,\ x \in X,\ \Phi^{-1}(b)x = g \text{ in } G. \end{array}$$

Proposition 7.1. *The system S above is convergent and defines the HNN-extension of G by by (A, B, Φ). Every irreducible normal form admits a unique decomposition as*

$$g = g_0 t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n$$

with n minimal such that $n \geq 0$, $g_0 \in G \setminus \{1\}$, and either $\varepsilon_i = -1$ with $g_i \in X$ or $\varepsilon_i = 1$ with $g_i \in Y$, for all $1 \leq i \leq n$.

Proof. Obviously, Γ^*/S defines the HNN-extension of G by by (A, B, Φ). Although the system has length-increasing rules it is not too difficult to prove termination. Local confluence is straightforward, so S is indeed convergent. Since all elements of G are irreducible we see that G embeds into the HNN-extension. Moreover, it is also clear that we obtain the normal form as stated in the proposition. □

This convergent system also leads to the following well-known classical fact.

Corollary 7.2. *Assume that have the following properties: H is finitely generated and has a decidable word problem, membership problems for A and B are solvable, and the isomorphism $\Phi : A \to B$ is effectively calculable. Then the HNN-extension of G by by (A, B, Φ) has a decidable word problem.*

Proof. We may represent all group elements in H by length-lexicographic first elements (i.e., choose among all geodesics the lexicographical first one). The transversal X (resp. Y) may be chosen to consist of the length-lexicographic first element of each coset Ag (resp. Bg), where g runs over G. Given g we can compute the representative of Ag in X (resp. Bg in Y), because membership is decidable for A and B. Now, given $b \in B$, the ability to compute Φ allows us to find $a \in A$ with $\Phi(a) = b$. Thus, all steps in computing normal forms are effective. □

It should be clear however that the purpose of the system S above is not to decide the word problem effectively; but rather to facilitate straightforward proofs of other results, such as Britton's lemma. Consider the following system B of Britton reduction rules.

$$\begin{array}{lll} t^{-1}t & \longrightarrow & 1; \qquad tt^{-1} \longrightarrow 1; \quad gh \longrightarrow f, \text{ if } gh = f \text{ in } G; \\ t^{-1}at & \longrightarrow & \Phi(a) \quad \text{if } a \in A; \\ tbt^{-1} & \longrightarrow & \Phi^{-1}(b) \quad \text{if } b \in B \end{array}$$

The system B is length reducing, but not confluent. However, $\underset{B}{\Longrightarrow} \subseteq \overset{*}{\underset{H}{\Longrightarrow}}$, hence we can think of B as as subsystem of H. Britton's lemma says that B is confluent on all words which represent 1 in the HNN-extension. Here is a proof using our system S. Consider any Britton reduced word g. It has the form $g = g_0 t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n$. Applying rules from H does not destroy the property of being Britton reduced and neither t nor t^{-1} can vanish. Thus, if g reduces to the empty word using H, then g is already the empty word.

Observe that B is not a geodesic system, because $at\Phi(a)^{-1}$ is Britton reduced, but $at\Phi(a)^{-1} = t$. In Example 8.2 below we construct a geodesically perfect rewriting system for an HNN-extension.

7.4 Free products with amalgamation

There is a natural convergent (resp. geodesically perfect) rewriting system which defines amalgamated products. Let A and B be groups intersecting in a common subgroup H. This time we choose transversals for cosets of H in A and in B; that is $X \subseteq A$ and $Y \subseteq B$ with $1 \in X \cap Y$ such that there are unique decompositions $A = HX$ and $B = HY$. We let $\Gamma = (A \cup B) \setminus \{1\}$ and we identify 1 with the empty word in Γ^*.

We use the convention to write $[ab]$ for the product ab whenever it is defined. This means $[ab]$ is viewed as a letter in Γ or $[ab] = 1$ and it is defined if either $a, b \in A$ or $a, b \in B$.

The system $S \subseteq \Gamma^2 \times (\{1\} \cup \Gamma \cup \Gamma^2)$ is now defined by the following rules:

$$\begin{array}{lll} ab & \longrightarrow & [ab] \quad \text{if } [ab] \text{ is defined,} \\ ab & \longrightarrow & [ah]y \quad \text{if } 1 \neq a \in A,\ h \in H,\ b \neq y \in Y, \text{ and } b = [hy] \\ ba & \longrightarrow & [bh]x \quad \text{if } 1 \neq b \in B,\ h \in H,\ a \neq x \in X, \text{ and } a = [hx] \end{array}$$

The system defines the amalgamated product $G = A *_H B$. It is terminating by a length lexicographical ordering. Local confluence follows by a direct inspection, whence convergence. Again we obtain the normal form theorem (*cf.* [36, Corollary 4.4.1]): every element g of G has a unique decomposition as

$$g = [hg_0]g_1 \cdots g_n,$$

where $h \in H$, g_i is a non-trivial element of $X \cup Y$ and g_i and g_{i+1} do not lie in the same factor. However, in practice we may not wish to compute transversals explicitly. So let us apply only length reducing rules $ab \longrightarrow [ab]$ only until we end up with a word $g = g_0 \cdots g_n$, to which no length reducing rule may be applied. Since we cannot apply length reducing rules to g we obtain that

$$\forall 0 \leq i < n : \; g_i \in A \iff g_{i+1} \in B \setminus H \;\wedge\; g_i \in B \iff g_{i+1} \in A \setminus H.$$

Further applications of the rules of S preserve this property. Thus, S is geodesically perfect, even if we use the length preserving rules only in the direction indicated above. Moreover, if we cannot apply length reducing rules to $g = g_0 \cdots g_n$ then we have $g = 1$ if and only if both $n = 0$ and $g_0 = 1$.

8 Stallings' pregroups and their universal groups

We now turn to the notion of pregroups in the sense of Stallings, [49], [50]. A *pregroup* P is a set P with a distinguished element ε, equipped with a partial multiplication $m : D \to P$, $(a, b) \mapsto ab$, where $D \subseteq P \times P$, and an involution (or *inversion*) $i : P \to P$, $a \mapsto a^{-1}$, satisfying the following axioms for all $a, b, c, d \in P$. (By "ab is defined" we mean to say that $(a, b) \in D$ and $m(a, b) = ab$.)

(P1) $a\varepsilon$ and εa are defined and $a\varepsilon = \varepsilon a = a$;

(P2) $a^{-1}a$ and aa^{-1} are defined and $a^{-1}a = aa^{-1} = \varepsilon$;

(P3) if ab is defined, then so is $b^{-1}a^{-1}$, and $(ab)^{-1} = b^{-1}a^{-1}$;

(P4) if ab and bc are defined, then $(ab)c$ is defined if and only if $a(bc)$ is defined, in which case
$$(ab)c = a(bc);$$

(P5) if ab, bc, and cd are all defined then either abc or bcd is defined.

It is shown in [29] that (P3) follows from (P1), (P2), and (P4), hence can be omitted.

The *universal group* $U(P)$ of the pregroup P can be defined as the quotient monoid

$$U(P) = \Gamma^* / \{\, ab = c \mid m(a, b) = c \,\},$$

where $\Gamma = P \setminus \{\varepsilon\}$ and $\varepsilon \in P$ is identified again with the empty word $1 \in \Gamma^*$. The elements of $U(P)$ may therefore represented by finite sequences $(a_1, \ldots, a_n)$ of elements from Γ such that $a_i a_{i+1}$ is not defined in P for $1 \leq i < n$: such sequences are called P-*reduced* sequences or *reduced* sequences. Since every element in $U(P)$ has an inverse, it is clear that $U(P)$ forms a group.

If Σ is any set, then the disjoint union $P = \{\varepsilon\} \cup \Sigma \cup \overline{\Sigma}$ where $\overline{\Sigma}$ is a copy of Σ yields a pregroup with involution given by $\overline{\varepsilon} = \varepsilon$, $\overline{\overline{a}} = a$, for all $a \in \Sigma$, such that $p\overline{p} = \varepsilon$, for all $p \in P$. In this case the universal group $U(P)$ is nothing but the free group $F(\Sigma)$.

The universal property of $U(P)$ holds trivially, namely the canonical morphism of pregroups $P \to U(P)$ defines the left-adjoint functor to the forgetful functor from groups to pregroups.

Stallings [49] showed that composition of the inclusion map $P \to P^*$ with the standard quotient map $P^* \to U(P)$ is injective, where P^* is the free monoid on P. The first step of his proof establishes *reduced forms* of elements of $U(P)$, up to an equivalence relation $\approx$ which, for completeness, we describe here. Define first a binary relation $\sim$ on the set of finite sequences of elements of P by

$$(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) \sim (a_1, \ldots, a_i c, c^{-1} a_{i+1}, \ldots, a_n),$$

provided $(a_i, c), (c^{-1}, a_{i+1}) \in D$. Then Stallings' equivalence relation $\approx$ is the transitive closure of $\sim$.

Guiding examples are again amalgamated products and HNN-extensions.

Example 8.1. *As in Section 7.4, let A and B be groups intersecting in a common subgroup H. Consider the subset $P = A \cup B \subseteq G = A *_H B$. Define a partial multiplication $p \cdot q$ in the obvious way; that is $p \cdot q$ is defined if and only if either p, q are both in A or p, q both in B. Then P is a pregroup where $D = A \times A \cup B \times B$. We obtain the following geodesically perfect rewriting system (where the length is computed w.r.t. P, thus elements of P are viewed as letters).*

$$\begin{array}{rcll} 1 & \longrightarrow & \varepsilon & \\ p \cdot q & \longrightarrow & r & \text{if } (p,q) \in D, pq = r \in G \\ a \cdot b & \longleftrightarrow & ah \cdot h^{-1}b & \text{if } a \in A \setminus H,\ b \in B \setminus H. \end{array}$$

Example 8.2. *Let H be the HNN-extension HNN$(G; A, B, \Phi)$ as defined in Section 7.3 and, as before, let X and Y be transversals for A and B in G with $X \cap Y = \{1\}$. Consider the subset*

$$P = G \cup GtY \cup Gt^{-1}X \subset H.$$

We define a partial multiplication by the obvious rules (left to the reader) *according to the following table.*

$$\begin{array}{rll} G \times G & \longrightarrow G & \\ G \times GtY & \longrightarrow GtY & \\ G \times Gt^{-1}X & \longrightarrow Gt^{-1}X & \\ GtY \times G & \longrightarrow GtY & \\ Gt^{-1}X \times G & \longrightarrow Gt^{-1}X & \\ Gt^{-1}X \times GtY & \longrightarrow G & \textit{if the inner part } XG \textit{ is in } A \\ GtY \times Gt^{-1}X & \longrightarrow G & \textit{if the inner part } YG \textit{ is in } B. \end{array}$$

This defines a pregroup P for H, where

$$D = G \times G \cup G \times GtY \cup G \times Gt^{-1}X \cup GtY \times G \cup Gt^{-1}X \times G \cup S,$$

where S is the subset of $Gt^{-1}X \times GtY \cup GtY \times Gt^{-1}X$ where inner parts XG or YG belong to A or B, as appropriate. The partial multiplication table can be directly read from the convergent system we used in Section 7.3. As we shall see below, it defines an (infinite) geodesically perfect rewriting system, where again we view elements of P as letters. Note also that we could replace X and Y by $X = Y = G$ throughout the definition of our pregroup P in which the multiplication table could be slightly more simply described, but would be unnecessarily large.

In [49] an alternative pregroup for H is defined with underlying set consisting of equivalence classes of elements of $G \cup t^{-1}G \cup Gt \cup t^{-1}Gt$ under the equivalence relation generated by $t^{-1}at \sim \Phi(a)$, for $a \in A$. However we feel that the resulting rewriting rules are obscured by the equivalence relation on the underlying set.

The following is the principal result on the universal groups of pregroups.

Theorem 8.3 (Stallings [49])**.** *Let P be a pregroup. Then:*

1) *Every element of $U(P)$ can be represented by a P-reduced sequence;*

2) *any two P-reduced sequences representing the same element are $\approx$ equivalent, in particular they have the same length;*

3) *P embeds into $U(P)$.*

8.1 Rewriting systems for universal groups

The result of [49] cited above may be regarded as showing that composition of the inclusion map $P \to P^*$ with the standard quotient map $P^* \to U(P)$ is injective, where P^* is the free monoid on P. We show here how to achieve this with the help of a geodesically perfect Thue system. Since this approach may be new we work out the details.

It is convenient to work over P^* and view each element of P as a letter. We have to distinguish whether a product is taken in the free monoid P^* or in P, and we introduce the following convention. Whenever we write $[ab]$ we mean that $(a,b) \in D \subseteq P \times P$ with $m(a,b) = [ab] \in P$: that is the product ab is defined in P and yields a letter.

The system $S = S(P) \subseteq P^* \times P^*$ is now defined by the following rules.

$$\begin{array}{rcll} \varepsilon & \longrightarrow & 1 & \text{(= the empty word)} \\ ab & \longrightarrow & [ab] & \text{if } (a,b) \in D \\ ab & \longleftrightarrow & [ac][c^{-1}b] & \text{if } (a,c),\, (c^{-1},b) \in D \end{array}$$

Theorem 8.4. *Let P be a pregroup. Then the following hold.*

1) $P^*/S(P) \simeq U(P)$.

2) S is a geodesically perfect Thue system.

Proof. Obviously, P^*/S defines $U(P)$ which proves 1). To prove 2) we show first that the system S is strongly confluent. For this we have to consider two rules such that the left-hand sides overlap. Strong confluence involving only symmetric rules is trivial. Thus, we may assume that one rule is length-reducing. If one of the rules is $\varepsilon \longrightarrow 1$, then (by symmetry) the other rule is either $\varepsilon b \longrightarrow b$ or $\varepsilon b \longrightarrow c[c^{-1}b]$. Since $(c^{-1},b) \in D$ implies $(c, c^{-1}b) \in D$ and $[c(c^{-1}b)] = b$ [49], both situations lead to b in at most one step. The next situation is:

$$[ab] \underset{S}{\Longleftarrow} ab \underset{S}{\Longrightarrow} [ac][c^{-1}b]$$

Since (a,b) and (c^{-1},b) both belong to D we have $(a, c(c^{-1}b)) \in D$, as above, and (P4) implies that $(ac, c^{-1}b) \in D$, so we can apply the rule $[ac][c^{-1}b] \longrightarrow [ab]$. Finally, we have to consider:

$$yd \underset{S}{\Longleftarrow} abd \underset{S}{\Longrightarrow} az$$

with $a,b,d \in P$ and $y,z \in P^*$. We may assume that one rule is length-reducing of type $ab \longrightarrow y = [ab]$. The other rule is either of type $bd \longrightarrow [bd]$ or of type $bd \longleftrightarrow [bc][c^{-1}d]$. Assume first that $(b,d) \in D$, then in both case we can use:

$$[ab]d \underset{S}{\Longrightarrow} [abb^{-1}][bd] \;= a[bd] \underset{S}{\Longleftarrow} a[bc][c^{-1}d].$$

The remaining case is that the $(b,d) \notin D$ and the situation is:

$$[ab]d \underset{S}{\Longleftarrow} abd \underset{S}{\Longrightarrow} a[bc][c^{-1}d].$$

Since (a,b), (b,c) and $(c,c^{-1}d)$ are in D, (P5) implies that either abc or $bcc^{-1}d = bd$ is defined in P. But bd is not defined, therefore abc is defined. We obtain:

$$[ab]d \underset{S}{\Longrightarrow} [abc][c^{-1}d] \underset{S}{\Longleftarrow} a[bc][c^{-1}d].$$

Now we show that S is geodesic, from which it follows that it is geodesically perfect. Start with a sequence $w \in P^*$ and apply only length-reducing rules until this in no longer possible. Clearly, the resulting sequence is P-reduced: $w \overset{*}{\underset{S}{\Longrightarrow}} a_1 \cdots a_n \in \Gamma^*$ such that $a_i a_{i+1}$ is not defined in P for $1 \leq i < n$. Possibly, one can still apply the symmetric rules, but we claim that any application of the symmetric rules gives again a P-reduced system. Indeed, assume $u \in \Gamma^*$ is P-reduced, but it is not P-reduced after one application of a length-preserving rule from $S(P)$. Then there are four consecutive elements $abde$ in u and an element $c \in P$ such that neither ab nor bd nor de is defined, but bc, $c^{-1}d$ are defined and either $a(bc)$ or $(bc)(c^{-1}d)$ or $(c^{-1}d)e$ is defined. Assume the product $a(bc)$ is defined. Then the sequence a, bc, c^{-1}, d satisfies the premise of the axiom (P5), so either $a(bc)c^{-1} = ab$ or $(bc)c^{-1}d = bd$ must be defined, contradicting the assumption that u is P-reduced. Similarly, $(c^{-1}d)e$ cannot be defined. Suppose now that $(bc)(c^{-1}d)$ is defined. Then the sequence $b, c, c^{-1}d$ satisfies the premise of (P4), since (bc) and $c(c^{-1}d)$ are defined. Since $(bc)(c^{-1}d)$ is defined (P4) implies that $b(c^{-1}(cd)) = b(1d) = bd$ is defined, in contradiction with P-reducibility of u. □

Remark 8.5. *Stallings' normal form theorem 8.3 is now a consequence of Theorem 8.4 because elements from P are irreducible, and the rewriting system is geodesically perfect. Thus, P-reduced sequences that define the same elements in $U(P)$ are $\approx$ equivalent.*

Remark 8.6. *As above let $\Gamma = P \setminus \{\varepsilon\}$. Since $S(P) \subseteq P^* \times P^*$ is strongly confluent and geodesic, we obtain a geodesically perfect presentation of the universal group $U(P)$. In some sense it is however nicer to have such a presentation over Γ. So, let us put $S'(P) \subseteq \Gamma^* \times \Gamma^*$ defined by the following rules:*

$$\begin{array}{rcll} aa^{-1} & \longrightarrow & 1 & \text{if } a \in \Gamma \\ ab & \longrightarrow & c & \text{if } (a,b) \in D,\ a \neq b^{-1},\ [ab] = c \\ ab & \longleftrightarrow & [ac][c^{-1}b] & \text{if } (a,c),\ (c^{-1},b) \in D \end{array}$$

The difference is that a rule $aa^{-1} \longrightarrow \varepsilon \in S$ ($\varepsilon \in P$ is a letter) is replaced by $aa^{-1} \longrightarrow 1 \in S'(P)$. This rule of $S'(P)$ needs two steps of $S(P)$, but in $S(P)$ we win strong confluence, whereas $S'(P)$ is not strongly confluent. However confluence of $S(P)$ transfers to $S'(P)$. Hence, both systems $S(P)$ and $S'(P)$ are geodesically perfect.

Using the geodesically perfect system $S(P)$ for $U(P)$ where we P is finite we see that the result of Rimlinger [44] leads to the following statement which is slightly stronger than the result of [24].

Corollary 8.7. *Let G be a finitely generated group. The following conditions are equivalent.*

1.) G is virtually free.

2.) G can be presented by some finite geodesically perfect system.

3.) G can be presented by some finite geodesic system.

Proof. By Rimlinger [44], a finitely generated virtually free group is the universal group $U(P)$ of some finite pregroup P. By Theorem 8.4 it has a presentation by the (finite) geodesically perfect system $S(P)$. In our setting every geodesically perfect system is geodesic, so we get the implication from 2.) to 3.) for free. In order to pass from 3.) to 1.) one has to show that the set of words which are equivalent to $1 \in G$ forms a context-free language. This is can be demonstrated using an argument from [15], which has also been used in [24]. Consider a word w and write it as as $w = uv$ such that u is geodesic. The prefix u is kept on a push down stack. Suppose that $v = av'$, for some letter a. Push a onto the top of the stack: so the stack becomes ua. There is no reason to suppose that ua is geodesic and we perform length reducing reduction steps on it to produce an equivalent geodesic word $\widehat{u}$. Suppose this requires k steps:

$$ua \underset{S_R}{\overset{k}{\Longrightarrow}} \widehat{u}$$

Let us show that we can bound k by some constant depending on only on S. Indeed for all letters a we may fix a word w_a such that $aw_a \underset{S_R}{\overset{*}{\Longrightarrow}} 1$. But this means

$$\widehat{u}w_a \underset{S_R}{\overset{*}{\Longrightarrow}} \widetilde{u},$$

where $\widetilde{u}$ is geodesic and $\widetilde{u}$ represents the same group element as u did. But u was geodesic, too. Hence $|u| = |\widetilde{u}|$. Therefore $|\widehat{u}| \geq |u| - |w_a|$ and this tells us $k \leq |w_a|$. Since k is bounded by some constant we see that the whole reduction process involves a bounded suffix of the word ua, only. This means we can factorise $ua = pq$ and $\widehat{u} = pr$, where the length of q is bounded by some constant depending on S only. Moreover, $q \underset{S_R}{\overset{k}{\Longrightarrow}} r$. Since the length of q is bounded this reduction can be performed using the finite control of the pushdown automaton. The automaton stops once the input has been read and then the stack gives us a geodesic corresponding to the input word w. In particular, the set of words which represent 1 in the group is context-free. Thus, the group presented is context-free; and using a result of Muller and Schupp [37] we see that G is virtually free. □

8.2 Characterisation of pregroups in terms of geodesic systems

In this section we consider Thue systems $S \subseteq \Gamma^* \times \Gamma^*$, corresponding to group presentations, i.e., $\Gamma = X \cup X^{-1}$ and S contains all the rules $xx^{-1} \to 1, x^{-1}x \to 1, x \in X$. We shall refer to these as *group rewriting systems*. We say that a rewriting system $S \subseteq \Gamma^* \times \Gamma^*$ is *triangular* if each rule $\ell \to r \in S$ satisfies the "triangular" condition: $|\ell| = 2$, $|r| \leq 1$, so every rule in S is of the form $ab \to c$ where $a, b \in \Gamma$ and $c \in \Gamma \cup \{1\}$. Observe that a triangular system is length-reducing.

We also say that S is *almost triangular* if $S = S' \cup S^\circ$, where S' is triangular and all rules in S° are trivial, i. e., of the form $a \to 1$, for some $a \in \Gamma$. Non-trivial examples of triangular systems come from triangulated presentations of groups. Namely, if $\langle X \mid R \rangle$ is a presentation of a group then one can *triangulate* this presentation by adding new generators and replacing old relations by finitely many triangular ones.

Another type of example arises from pregroups. Let P be a pregroup. In Section 8.1 we defined two rewriting systems $S(P)$ and $S'(P)$ associated with P that define the universal group $U(P)$. Notice that the length-reducing part $S'(P)_R$ of $S'(P)$ is triangular (here $\Gamma = P \setminus \{\varepsilon\}$):

$$S'(P)_R = \{aa^{-1} \to 1, ab \to c \mid a, b, c \in \Gamma, (a, b) \in D, [ab] = c, a \neq b^{-1}\},$$

meanwhile, the length reducing part $S(P)_R$ of $S(P)$ is almost triangular, since it contains the trivial rule $\varepsilon \to 1$.

Theorem 8.4 implies the following result.

Corollary 8.8. *Let P be a pregroup. Then $S'(P)_R$ is a triangular geodesic system, $S(P)$ is an almost triangular geodesic system and $U(P) = \Gamma^*/S'(P)_R = P^*/S(P)_R$.*

Proof. It suffices to observe, that $S'(P)_R$ and $S'(P)$ define the same equivalence relation on Γ^*. Indeed, every rule of the type $ab \to [ac][c^{-1}b]$, where $[ac]$ and $[c^{-1}b]$ are defined, can be realized as a following rewriting sequence in $S'(P)_R$:

$$ab \leftarrow acc^{-1}b \to [ac]c^{-1}b \to [ac][c^{-1}b],$$

which shows that $S'(P)_R$ and $S'(P)$ are equivalent. The rest follows from Theorem 8.4 and Remark 8.6. □

To prove the converse of this corollary we need some notation. Let $S \subseteq \Gamma^* \times \Gamma^*$ be a triangular group rewriting system, where $\Gamma = X \cup X^{-1}$. The congruence $\overset{*}{\underset{S}{\Longleftrightarrow}}$ on Γ^* induces an equivalence relation on the subset $\Gamma \cup \{1\}$, which we denote by $\approx$. Define P_S to be the quotient $(\Gamma \cup \{1\})/\approx$ and write $[z]$ for the equivalence class of the element $z \in \Gamma \cup \{1\}$ and in addition ε for the equivalence class of 1. Define an involution $p \to p^{-1}$ on P_S by setting $[x]^{-1} = [x^{-1}]$ and $[x^{-1}]^{-1} = [x]$, for $x \in X$, and setting $\varepsilon^{-1} = \varepsilon$. (Note that, since S is a group rewriting system, $x \approx 1$ if and only if $x^{-1} \approx 1$, so this involution is well defined.) Now we define a "partial multiplication" on P_S as follows.

- For $p, q \in P_S \setminus \{\varepsilon\}$ the product pq is defined and equal to s if there exist $x, y \in \Gamma$ such that $p = [x]$, $q = [y]$ and there is a rule $xy \to z \in S$, with $z \in \Gamma \cup \{1\}$ and $s = [z]$.
- For all $p \in P_S$ we put $p\varepsilon = \varepsilon p = p$ and $pp^{-1} = p^{-1}p = \varepsilon$.

It is not hard to see that the partial multiplication on P_S is well-defined.

Lemma 8.9. *Let S be a geodesic triangular group rewriting system. Then:*

1) P_S is a pregroup.

2) $U(P_S)$ is isomorphic to the group Γ^*/S.

Proof. Clearly, the axioms P1) and P2) hold in P_S by construction. It suffices to show that P4) and P5) hold in P_S, in which case P3) follows.

Checking P4). If any one of $p, q, r = \varepsilon$ then P4) holds trivially, so we may assume that $p, q, r \in P_S \setminus \{\varepsilon\}$. Suppose then $p = [a], q = [b], r = [c] \in P_S$ and the products pq and qr are defined, i.e., S contains rules $ab \to x$ and $bc \to y$ for some $x, y \in \Gamma \cup \{1\}$. Suppose also that $(pq)r$ is defined in P_S, so either $[x] = [z]$ and $zc \to t \in S$ for some $z, t \in \Gamma \cup \{1\}$, or $[x] = \varepsilon$, in which case let us define $t = c$. This means that $abc \overset{*}{\underset{S}{\Longleftrightarrow}} t$, for some $t \in \Gamma \cup \{1\}$ and also $abc \overset{*}{\underset{S}{\Longleftrightarrow}} yc$. As S is geodesic either S contains a rule $yc \to u$, for some $u \in \Gamma \cup \{1\}$, or $y = 1$, in which case let us define $u = c$. Then $(pq)r = [t] = [u] = p(qr)$ in P_S. It follows, by symmetry, that P4) holds.

P5). Again we may assume we have $p, q, r, s \in P_S \setminus \{\varepsilon\}$ such that $p = [a], q = [b], r = [c], s = [d]$ and the products pq, qr, rs are defined; so there are rules $ab \to x, bc \to y, cd \to z \in S$. We need to show that either pqr or qrs is defined. Assume pqr is not defined. This means in particular that $y \neq 1$ and that S contains no rule with left hand side ay.

We may rewrite $abcd$ in two different ways: $abcd \to xcd \to xz$ and $abcd \to ayd$. As S is geodesic either S must contain a rule which can be applied to ayd or one of a, y, d must be 1. Given our assumptions this means that S contains a rule with left hand side yd. Thus we have $(qr)s$ defined, so P5) holds. This proves the first statement.

The second statement follows from Theorem 8.4, Remark 8.6 and Corollary 8.8. Indeed, it suffices to note that, by construction, the system S is the length reducing part of the system $S'(P_S)$ associated with the pregroup P_S. □

Combining Corollary 8.8 and Lemma 8.9 one gets the following characterisation of pregroups and their universal groups in terms of triangular geodesic systems.

Theorem 8.10. *Let P be a pregroup. Then the reduced part of the rewriting system $S'(P)$ is a geodesic triangular group system which defines the universal group $U(P)$. Conversely, if S is a triangular geodesic group system then P_S is a pregroup, whose universal group is that defined by S.*

This result gives a method of constructing a potentially useful pregroup for a group given by a presentation in generators and relators. It would be helpful to have a KB like procedure for finding such pregroups.

Problem 8.11. *Design an (KB-like) algorithm that for a given finite triangular rewriting system finds an equivalent triangular geodesic system.*

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