From local to global temporal logics over Mazurkiewicz traces

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The paper is dedicated to the 60th birthday of Christian Choffrut

Abstract

We review some results on global and local temporal logic on Mazurkiewicz traces. Our main contribution is to show how to derive the expressive completeness of global temporal logic with respect to first order logic [9] from the similar result on local temporal logic [11].

Key words: Temporal logics, Mazurkiewicz traces, concurrency

1 Introduction

Trace theory has a long history in computer science. It started with the classical works of Keller [16] and Mazurkiewicz [17,18]. Contributions to trace theory include combinatorial properties, formal languages, automata, and logic. Christian Choffrut participated in the development of trace theory in all these areas with papers [2–4,6–8] and his survey in [5]. In the present paper we focus on linear temporal logics which have received quite an attention, see [1,19–27]. In [9] we have shown that a pure future linear temporal logic is powerful enough to express all first-order properties, if a global semantics is used, where

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the formulae are interpreted at configurations. This means traces are viewed as labelled partial orders and the interpretation is done along cuts as e.g. in [25]. The drawback of this approach is however that the satisfiability problem and the model checking problem become non-elementary, [26]. On the other hand, a local interpretation of linear temporal logics at vertices leads to polynomially space bounded satisfiability and model checking algorithms. In fact, all local temporal logics over traces where the modalities are definable in monadic second order logic are decidable in PSPACE [14]. Therefore local logics are much more of interest. In [11] we were able to prove that a pure future local linear temporal logic is also powerful enough to express all first-order properties.

The main contribution here is a direct translation of local temporal logic formulae into equivalent global temporal logic formulae. From this translation, using the expressive completeness of local temporal logics with respect to first order logic ([11]) we get as a corollary the expressive completeness of global temporal logics, which is the main result of [9]. Actually, we strengthen the result of [9] in two respects. First we show that we can use the basic *next* modality ($\mathsf{EX}\,\varphi$) instead of the more precise $\langle a \rangle \varphi$ modality to get the expressive completeness of global temporal logics. Second, we show that this expressive completeness can be obtained using *robust* global formulae whose evaluation do not depend on whether the semantics of *until* allows a finite or an infinite prefix in the factorization.

2 Mazurkiewicz traces

We recall some standard notations from trace theory which will be used in the paper. A *dependence alphabet* is a pair (Σ, D) where the alphabet Σ is a finite set and the *dependence relation* $D \subseteq \Sigma \times \Sigma$ is reflexive and symmetric. The *independence relation* I is the complement of D. For $A \subseteq \Sigma$, the set of letters dependent on A is denoted by $D(A) = \{b \in \Sigma \mid (a, b) \in D \text{ for some } a \in A\}$.

A *Mazurkiewicz trace* is an equivalence class of a labelled partial order $t = [V, \leq, \lambda]$ where V is a set of vertices labelled by $\lambda : V \to \Sigma$ and $\leq$ is a partial order over V satisfying the following conditions: For all $x \in V$, the downward set $\downarrow x = \{y \in V \mid y \leq x\}$ is finite, and for all $x, y \in V$ we have that $(\lambda(x), \lambda(y)) \in D$ implies $x \leq y$ or $y \leq x$, and that $x \lessdot y$ implies $(\lambda(x), \lambda(y)) \in D$, where $\lessdot \, = \, < \setminus <^2$ is the immediate successor relation in t. For $x \in V$, we also define $\uparrow x = \{y \in V \mid x \leq y\}$ and $\Uparrow x = \{y \in V \mid x < y\}$.

The trace t is finite if V is finite and we denote the set of finite traces by $\mathbb{M}(\Sigma, D)$ (or simply $\mathbb{M}$). By $\mathbb{R}(\Sigma, D)$ (or simply $\mathbb{R}$), we denote the set of finite or infinite traces (also called *real traces*). We write $\mathrm{alph}(t) = \lambda(V)$ for the alphabet of t and we let $\mathrm{alphinf}(t) = \{a \in \Sigma \mid \lambda^{-1}(a) \text{ is infinite}\}$ be the set of letters occuring infinitely often in t.

We define the concatenation for traces $t_1 = [V_1, \leq_1, \lambda_1]$ and $t_2 = [V_2, \leq_2, \lambda_2]$, provided $\mathrm{alphinf}(t_1) \times \mathrm{alph}(t_2) \subseteq I$. It is given by $t_1 \cdot t_2 = [V, \leq, \lambda]$ where V is the disjoint union of V_1 and V_2, $\lambda = \lambda_1 \cup \lambda_2$, and $\leq$ is the transitive closure of the relation $\leq_1 \cup \leq_2 \cup (V_1 \times V_2 \cap \lambda^{-1}(D))$. The set $\mathbb{M}$ of finite traces is then a monoid with the empty trace $1 = (\emptyset, \emptyset, \emptyset)$ as unit. If we can write $t = rs$, then r is a prefix of t and s is a suffix of t.

We denote by $\min(t)$ the set of minimal vertices of t. We let $\mathbb{R}^1 = \{t \in \mathbb{R} \mid |\min(t)| = 1\}$ be the set of traces with exactly one minimal vertex.

We also use $\min(t)$ for the set $\lambda(\min(t))$ of labels of the minimal vertices of t, and similarly for $\max(t)$. What we actually mean is always clear from the context. If $t = [V, \leq, \lambda] \in \mathbb{R}$ is a real trace and $x \in V$ is a vertex then we also write $x \in t$ instead of $x \in V$.

A trace p is called a *prime*, if it is finite and has a unique maximal element. The set of all primes in $\mathbb{R}$ is denoted by $\mathbb{P}$. We have $\mathbb{P} \subseteq \mathbb{M}$, whereas $\mathbb{R}^1$ contains infinite traces (if $\Sigma \neq \emptyset$).

3 Local temporal logics

The syntax of the basic linear temporal logic LTL_Σ is given by

$$\varphi ::= \top \mid a \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{EX}\,\varphi \mid \varphi\,\mathsf{U}\,\varphi.$$

where a ranges over Σ and $\top$ denotes *true*. As ususal, we use $\mathsf{F}\,\varphi$ (*future* or *eventually* φ) as an abbreviation for $\top\,\mathsf{U}\,\varphi$ and $\mathsf{G}\,\varphi = \neg\,\mathsf{F}\,\neg\varphi$ (*globally* in the sense of *always* φ).

Here EX denotes the usual (existential) *next*-operator and U means *until*. For (non-empty) finite or infinite words there is a standard semantics, and we have the following classical results:

The notions of first-order definability, star-freeness, aperiodicity, and LTL_Σ-definability lead to the same class of formal languages.

These results have been generalized in a sequence of papers to traces, [9,11–13,15,25]. However, the situation for traces is more complex. Traces are labelled partial orders and hence there are (at least) two natural semantics for LTL_Σ. We can define a *local* and a *global* semantics of linear temporal logics, and these semantics are quite different. In the following, we review some of our results and the reference logic for us is always first-order logic. We do not repeat its definition (which is the usual one and can be found e.g. in [13]), because we never work explicitely with it. Our results rather relate expressive powers of different temporal logics to each other.

3.1 Local semantics

Let $t = [V, \leq, \lambda] \in \mathbb{R}$ be a nonempty real trace and $x \in t$ be a vertex. The local semantics of LTL_Σ is defined by:

$$
\begin{array}{lll}
t, x \models \top & & \\
t, x \models a & \text{if} & \lambda(x) = a \\
t, x \models \neg\varphi & \text{if} & t, x \not\models \varphi \\
t, x \models \varphi \vee \psi & \text{if} & t, x \models \varphi \text{ or } t, x \models \psi \\
t, x \models \mathsf{EX}\,\varphi & \text{if} & \exists y\ (x \lessdot y \text{ and } t, y \models \varphi) \\
t, x \models \varphi \,\mathsf{U}\, \psi & \text{if} & \exists z\ (x \leq z \text{ and } t, z \models \psi \text{ and } \forall y\ (x \leq y < z) \Rightarrow t, y \models \varphi).
\end{array}
$$

Together with the local semantics LTL_Σ is denoted by $\mathrm{LocTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$ henceforth. This semantics is called *local* since a formula is evaluated at some vertex x of t which corresponds to the occurrence of a local event of the concurrent behavior represented by t.

Note that the temporal logics $\mathrm{LocTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$ is pure future, i.e., whether $t, x \models \varphi$ holds or not only depends on the suffix of t defined by $\uparrow x$ consisting of the events in t which are in the future of x. Formally, we have $t, x \models \varphi$ if and only if $\uparrow x, x \models \varphi$. Therefore, we could also define the semantics $t \models' \varphi$ for traces $t \in \mathbb{R}^1$ only omitting the vertex x which is implicitely the minimal vertex of t, i.e., $t \models' \varphi$ if and only if $t, \min(t) \models \varphi$ for $t \in \mathbb{R}^1$. For instance we would have $t \models' \varphi \,\mathsf{U}\, \psi$ if there exists $z \in t$ such that $\uparrow z \models' \psi$ and for all $y \in t$, $y < z$ implies $\uparrow y \models' \varphi$. We draw the attention to this alternative definition because the corresponding one will be more convenient for the global semantics. Hence this remark should help linking the two definitions.

In the following proofs we will use mainly some fragments of $\mathrm{LocTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$. To introduce these fragments, we first need some notations. For $x \in t$ and $c \in \Sigma$, we denote by x_c the unique minimal vertex of $\Uparrow x \cap \lambda^{-1}(c)$ if it exists, i.e., if $\Uparrow x \cap \lambda^{-1}(c) \neq \emptyset$. Note that $x < x_c$ if x_c exists.

We consider the local temporal logic $\mathrm{LocTL}_\Sigma[(\mathsf{X}_a \leq \mathsf{X}_b), \mathsf{XU}_a]$ the syntax of which is given by

$$\varphi ::= \top \mid a \mid (\mathsf{X}_a \leq \mathsf{X}_b) \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi\, \mathsf{XU}_a\, \varphi$$

where a, b range over Σ. The semantics of $\mathrm{LocTL}_\Sigma[(\mathsf{X}_a \leq \mathsf{X}_b), \mathsf{XU}_a]$ is defined as follows.

$$\begin{aligned}
t, x \models (\mathsf{X}_a \leq \mathsf{X}_b) \quad & \text{if} \;\; x_a, x_b \text{ exist and } x_a \leq x_b \\
t, x \models \varphi\, \mathsf{XU}_a\, \psi \quad & \text{if} \;\; \exists z\ (x < z \text{ and } \lambda(z) = a \text{ and } t, z \models \psi \text{ and} \\
& \qquad\qquad \forall y\ (x < y < z \text{ and } \lambda(y) = a) \Rightarrow t, y \models \varphi).
\end{aligned}$$

It is shown in [11] that $\mathrm{LocTL}_\Sigma[(\mathsf{X}_a \leq \mathsf{X}_b), \mathsf{XU}_a]$ is indeed a a fragment of $\mathrm{LocTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$. The main results of [11] can be stated now as follows:

Theorem 1 ([11]) *Let $L \subseteq \mathbb{R}$ be a first-order definable real trace language Then there are formulae $\varphi \in \mathrm{LocTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$ and $\psi \in \mathrm{LocTL}_\Sigma[(\mathsf{X}_a \leq \mathsf{X}_b), \mathsf{XU}_a]$ such that*

$$L \cap \mathbb{R}^1 = \{t \in \mathbb{R}^1 \mid t, \min(t) \models \varphi\} = \{t \in \mathbb{R}^1 \mid t, \min(t) \models \psi\}.$$

Theorem 2 ([11]) *Let $L \subseteq \mathbb{R}$ be a real trace language and let $\#$ be a new symbol ($\# \notin \Sigma$) which depends on all letters of Σ. Then the following assertions are equivalent.*

(1) The language L is first-order definable.
(2) There is a formula $\varphi \in \mathrm{LocTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$ such that

$$L = \{t \in \mathbb{R} \mid \#t, \# \models \varphi\}.$$

(3) There is a formula $\varphi \in \mathrm{LocTL}_\Sigma[(\mathsf{X}_a \leq \mathsf{X}_b), \mathsf{XU}_a]$ such that

$$L = \{t \in \mathbb{R} \mid \#t, \# \models \varphi\}.$$

Let $t \in \mathbb{R}$ be a possibly empty real trace. The global semantics of the linear temporal logic LTL_Σ is defined by:

$$
\begin{array}{lll}
t \models_g \top & & \\
t \models_g a & \text{if} & t \in a\mathbb{R} \\
t \models_g \neg\varphi & \text{if} & t \not\models_g \varphi \\
t \models_g \varphi \vee \psi & \text{if} & t \models_g \varphi \text{ or } t \models_g \psi \\
t \models_g \mathsf{EX}\,\varphi & \text{if} & \exists a \in \Sigma, s \in \mathbb{R} : \ t = as \text{ and } s \models_g \varphi \\
t \models_g \varphi \,\mathsf{U}\, \psi & \text{if} & \exists r \in \mathbb{M}, s \in \mathbb{R} : \ t = rs \text{ and } s \models_g \psi \text{ and} \\
& & \forall r', r'' \in \mathbb{R}, (r = r'r'' \text{ and } r'' \neq 1) \Rightarrow r''s \models_g \varphi.
\end{array}
$$

Analogously to the above, together with the global semantics we denote this logic by $\mathrm{GlobTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$ henceforth.

When the logic is not *pure future*, the global semantics must define more generally when $t, r \models_g \varphi$ where $t \in \mathbb{R}$ is a (possibly empty) real trace and r is a prefix of t corresponding to a partial execution. If s is the corresponding suffix then we have $t = rs$. Note that r may have several maximal events and s may have several minimal events. Hence the factorization $t = rs$ defines a *global cut* in the behavior t. This is why we call this semantics *global*.

Since our logic LTL_Σ is *pure future*, the truth value of $t, r \models_g \varphi$ only depends on the suffix s: formally, we have $t, r \models_g \varphi$ if and only if $s \models_g \varphi$. Since we only deal with pure future logics in this paper we have chosen to omit the prefix r simplifying the definition of the semantics.

For sake of completeness let us discuss a subtle point. It is not clear that the choice to define $t \models_g \varphi \,\mathsf{U}\, \psi$ as done above is the only natural one for traces. We required the first factor r to be finite, and we did it this way in order to find the usual semantics in case of infinite words, but as an alternative one could consider a semantics defined as follows:

$$
\begin{array}{lll}
t \models_g \varphi \,\mathsf{U}\, \psi & \text{if} & \exists r \in \mathbb{R}, s \in \mathbb{R} : \ t = rs \text{ and } s \models'_g \psi \text{ and} \\
& & \forall r', r'' \in \mathbb{R}, (r = r'r'' \text{ and } r'' \neq 1) \Rightarrow r''s \models'_g \varphi.
\end{array}
$$

We get stronger and more convenient results, if we do not need to pay attention what choice has been taken. To make this formal let us inductively define *robust* formulae.

$$\begin{array}{rl} \top & \text{is robust,} \\ a & \text{is robust (for } a \in \Sigma\text{),} \\ \neg\varphi & \text{is robust if } \varphi \text{ is robust,} \\ \varphi \vee \psi & \text{is robust if } \varphi \text{ and } \psi \text{ are robust,} \\ \mathsf{EX}\,\varphi & \text{is robust if } \varphi \text{ is robust,} \\ \varphi\,\mathsf{U}\,\psi & \text{is robust if } \varphi \text{ and } \psi \text{ are robust, and } \forall t \in \mathbb{R}, \\ & \qquad t \models_g \varphi\,\mathsf{U}\,\psi \Longleftrightarrow t \models'_g \varphi\,\mathsf{U}\,\psi. \end{array}$$

Note that $\mathsf{F}\,a$ is robust, whereas $\mathsf{F}\,\neg a$ is not robust since $a^\omega \models'_g \mathsf{F}\,\neg a$ but $a^\omega \not\models_g \mathsf{F}\,\neg a$. Actually, $t \models'_g \mathsf{F}\,\neg a$ for all trace $t \in \mathbb{R}$ whereas $t \models_g \mathsf{F}\,\neg a$ if and only if $t = a^k s$ for some $k \geq 0$ and $s \in \mathbb{R}$ with $a \notin \min(s)$.

Clearly, if φ is robust then $t \models_g \varphi$ if and only if $t \models'_g \varphi$.

In [9] another type of global formulae has been used. For $a \in \Sigma$ and a formula φ the global semantics of $\langle a \rangle\varphi$ (read next-a-φ) is defined by

$$t \models_g \langle a \rangle\varphi \;\text{ if }\; \exists s \in \mathbb{R} : \; t = as \text{ and } s \models_g \varphi.$$

It was not noticed in [9] that the modality $\langle a \rangle(-)$ can be expressed in $\mathrm{GlobTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$. So this is our first contribution.

Lemma 3 *A language $L \subseteq \mathbb{R}$ is expressible in* $\mathrm{GlobTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$ *if and only if it is expressible in* $\mathrm{GlobTL}_\Sigma[\langle a \rangle, \mathsf{U}]$.

Moreover, if φ is robust, then $\langle a \rangle\varphi$ can be expressed by some robust formula in $\mathrm{GlobTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$.

Proof. Obviously, EX can be defined by $\mathsf{EX}\,\varphi = \bigvee_{a\in\Sigma} \langle a \rangle\varphi$. For the other direction we show that the formula $\langle a \rangle\varphi$ can be expressed using EX and φ. In particular, $\langle a \rangle\varphi$ is replaced by a robust formula, if φ is robust.

First, for a given $\varphi \in \mathrm{GlobTL}_\Sigma$ let $m \in \mathbb{N}$ be such that for all $t \in \mathbb{R}$ and $a \in \Sigma$ we have $a^m t \models_g \varphi$ if and only if $a^{m+1} t \models_g \varphi$. The existence of m is ensured by the fact that $\{t \in \mathbb{R} \mid t \models_g \varphi\}$ is aperiodic by [13] (see also [10]). Here, we give a direct proof of this fact.

We define inductively $m(\varphi)$ by $m(\top) = 0$, $m(b) = 1$ for $b \in \Sigma$, $m(\neg\varphi) = m(\varphi)$, $m(\mathsf{EX}\,\varphi) = 1 + m(\varphi)$ and $m(\varphi \vee \psi) = m(\varphi\,\mathsf{U}\,\psi) = \max(m(\varphi), m(\psi))$. We show by structural induction on φ that for all $n \geq m(\varphi)$ and all $t \in \mathbb{R}$ and $a \in \Sigma$, we have $a^n t \models_g \varphi$ if and only if $a^{n+1} t \models_g \varphi$.

The result is clear for $\varphi = \top$. For $\varphi = b$ we use the fact that for $n \geq 1$, $a^n t \models_g b$ if and only if $a = b$ or $b \in \min(t)$ and $(a, b) \in I$. The induction is trivial for negation and disjunction. Assume now that $a^n t \models_g \mathsf{EX}\,\varphi$ with $n \geq 1 + m(\varphi)$. We have $a^n t = bs$ with $b \in \Sigma$ and $s \models_g \varphi$. If $a = b$ then $s = a^{n-1}t \models_g \varphi$ and we get $a^n t \models_g \varphi$ by induction. It follows that $a^{n+1}t \models_g \mathsf{EX}\,\varphi$. Now, if $a \neq b$ then $(a, b) \in I$ and we have $s = a^n r$ and $t = br$. By induction we get $a^{n+1}r \models_g \varphi$ and therefore $a^{n+1}t = ba^{n+1}r \models_g \mathsf{EX}\,\varphi$. Hence we have shown that $a^n t \models_g \mathsf{EX}\,\varphi$ implies $a^{n+1}t \models_g \mathsf{EX}\,\varphi$. The converse implication can be shown similarly.

Finally, assume that $a^n t \models_g \varphi \mathbin{\mathsf{U}} \psi$ and write $a^n t = rs$ with $s \models_g \psi$ and $r''s \models_g \varphi$ for all $r'r'' = r$ with $r'' \neq 1$. If a is independent of r then $s = a^n s'$ and we get by induction that $as = a^{n+1}s' \models_g \psi$ and $r''as = a^{n+1}r''s' \models_g \varphi$ for all $r'r'' = r$ with $r'' \neq 1$. Hence in this case $a^{n+1}t \models_g \varphi \mathbin{\mathsf{U}} \psi$. Now, if a depends on r then $a \leq r$ and $a^{n+1}t = (ar)s$. Let $r'r'' = ar$ with $r'' \neq 1$. If $r' = ar_1$ then $r = r_1 r''$ and we get $r''s \models_g \varphi$. Otherwise a is independent of r' and $r'' = ar_2$. We obtain $r = r'r_2$ and $a \leq r_2 \neq 1$. Therefore, $r_2 s \models_g \varphi$ and since $a^n \leq r_2 s$ we get also $r''s = ar_2 s \models_g \varphi$. Finally, we have shown $a^{n+1}t \models_g \varphi \mathbin{\mathsf{U}} \psi$. The converse can be shown similarly.

Now, for each $k \geq 1$ we define a macro a^k by

$$a^k = a \wedge \bigwedge_{1 \leq \ell < k} \neg \mathsf{EX}^{\ell} \neg a.$$

Note that a^k is a robust formula. We show that $t \models_g a^k$ if and only if $t \in a^k \mathbb{R}$ by induction on k. If $k = 1$ then the result is clear. Let now $k > 1$ and assume that $t \models_g a^k$. Note that $a^k = a^{k-1} \wedge \neg \mathsf{EX}^{k-1} \neg a$. By induction we deduce that $t = a^{k-1}s$ with $s \in \mathbb{R}$. If $s \notin a\mathbb{R}$ then $s \models_g \neg a$ and $t = a^{k-1}s \models_g \mathsf{EX}^{k-1} \neg a$, a contradiction. Hence $s \in a\mathbb{R}$ and $t \in a^k \mathbb{R}$.

Conversely, if $t \in a^k \mathbb{R}$ then we have $t \models_g a^{k-1}$ by induction and it remains to show that $t \models_g \neg \mathsf{EX}^{k-1} \neg a$. We have $t \models_g \mathsf{EX}^{k-1} \neg a$ if and only if there is a factorization $t = rs$ with $|r| = k - 1$ and $s \notin a\mathbb{R}$. This is indeed impossible if $t \in a^k \mathbb{R}$.

To conclude the proof of the lemma, we show that $\langle a \rangle \varphi$ is equivalent to the robust formula

$$(a^{m+1} \wedge \varphi) \vee \bigvee_{1 \leq k \leq m} (a^k \wedge \mathsf{EX}(\neg a^k \wedge \varphi)).$$

Assume first that $t \models_g \langle a \rangle \varphi$. If $t = a^{m+1}s \in a^{m+1}\mathbb{R}$ then we have $a^m s \models_g \varphi$ and we get $t = a^{m+1}s \models_g \varphi$ by definition of m. In this case, $t \models_g a^{m+1} \wedge \varphi$. Now, assume that $t = a^k s \in a^k \mathbb{R}$ for some $k \leq m$ and $s \in \mathbb{R} \setminus a\mathbb{R}$. Then $t \models_g a^k$ and $a^{k-1}s \models_g \varphi \wedge \neg a^k$.

Conversely, assume that $t \models_g a^{m+1} \wedge \varphi$ then $t = a^{m+1}s \models_g \varphi$ and by definition of m we get $a^m s \models_g \varphi$ and $t \models_g \langle a \rangle \varphi$. Finally, assume that $t \models_g a^k \wedge \mathsf{EX}(\varphi \wedge \neg a^k)$ for some $1 \leq k \leq m$. We have $t \in a^k\mathbb{R}$ and $t = bs$ with $b \in \Sigma$, $s \models_g \varphi$ and $s \notin a^k\mathbb{R}$. We deduce that $b = a$ and $t \models_g \langle a \rangle \varphi$. $\square$

With the help of Lemma 3 the main result of [9] can be stated as follows:

Theorem 4 ([9]) *A language $L \subseteq \mathbb{R}$ is first-order definable if and only if it is expressible in* $\mathrm{GlobTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$.

4 From local to global logic

The proofs in [9] and [11] are both very complex and technical. They are independent of each other, but one has the impression that the local result Theorem 2 is stronger than the global one in Theorem 4. We can confirm this impression by the following proposition. In the following # denotes again a new symbol (not in Σ) which depends on all letters of Σ.

Proposition 5 *Let $c \in \Sigma' = \Sigma \cup \{\#\}$ and $\varphi \in \mathrm{LocTL}_\Sigma[(\mathsf{X}_a \leq \mathsf{X}_b), \mathsf{XU}_a]$. Then we can effectively construct a* robust *formula $\overline{\varphi}^c \in \mathrm{GlobTL}_\Sigma[\langle a \rangle, \mathsf{U}]$ such that for all $t \in \mathbb{R}$ we have*

$$ct, \min_c(ct) \models \varphi \text{ if and only if } t \models_g \overline{\varphi}^c$$

where $\min_c(ct)$ is the minimal vertex of ct which is labelled c.

So, if we are willing to use Theorem 2, we get as a corollary a strengthening of Theorem 4 since the expressibility is obtained with *robust* formulae.

Corollary 6 *A language $L \subseteq \mathbb{R}$ is first-order definable if and only if it is expressible by some* robust *formula in* $\mathrm{GlobTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$.

Proof. Actually, we get as a corollary only the difficult part of the equivalence. So let $L \subseteq \mathbb{R}$ be first-order definable. By Theorem 2 we find a local formula $\varphi \in \mathrm{LocTL}_\Sigma[(\mathsf{X}_a \leq \mathsf{X}_b), \mathsf{XU}_a]$ such that

$$L = \{t \in \mathbb{R} \mid \#t, \# \models \varphi\}.$$

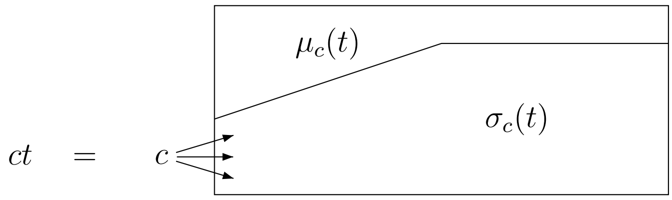
Let $\overline{\varphi}^{\#} \in \mathrm{GlobTL}_\Sigma[\langle a \rangle, \mathsf{U}]$ be the *robust* formula given by Proposition 5. We get

$$L = \{t \in \mathbb{R} \mid t \models_g \overline{\varphi}^{\#}\}.$$

It remains to apply Lemma 3 to get a *robust* formula from $\mathrm{GlobTL}_\Sigma[\mathsf{EX}, \mathsf{U}]$ defining L. $\square$

The remaining of this section is devoted to the proof of Proposition 5. The construction of $\overline{\varphi}^c$ is done by structural induction. Clearly, we have $\overline{\bot}^c = \bot$, $\overline{\neg\varphi}^c = \neg\overline{\varphi}^c$, and $\overline{\varphi \vee \psi}^c = \overline{\varphi}^c \vee \overline{\psi}^c$. For $a \in \Sigma$, we define $\overline{a}^c = \top$, if $a = c$, and $\overline{a}^c = \bot$, if $a \neq c$. The translation for $(\mathsf{X}_a \leq \mathsf{X}_b)$ and XU_a is much more involved and will use some auxiliary macros.

Since we deal with pure future logics, we have already noticed that whether $ct, \min_c(ct) \models \varphi$ only depends on the future of $\min_c(ct)$ in the trace ct. Hence, we define $\mu_c(t)$ and $\sigma_c(t)$ by the equation $t = \mu_c(t)\sigma_c(t)$ with c independent of $\mu_c(t)$ and $\min(\sigma_c(t)) \subseteq D(c)$. Note that $c\sigma_c(t) \in \mathbb{R}^1$ is the future of $\min_c(ct)$. Therefore, $ct, \min_c(ct) \models \varphi$ if and only if $c\sigma_c(t), \min(c\sigma_c(t)) \models \varphi$.



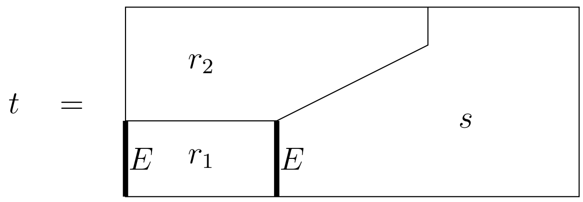
It is easier to give a formula for $(\mathsf{X}_a \leq \mathsf{X}_b)$ if the first a in t coincide with the first a in ct which is above $x = \min_c(ct)$ and similarly for b. This is the case if and only if $a, b \notin \mathrm{alph}(\mu_c(t))$. In order to reduce the general case to this simpler case, we will introduce a macro $(\mathrm{shift}_{c,E}(-))$ which allows to skip an arbitrary finite prefix of $\mu_c(t)$. In the special case of $(\mathsf{X}_a \leq \mathsf{X}_b)$ we will use this macro to skip all a's and b's contained in $\mu_c(t)$. Skipping a prefix that might be arbitrarily long requires some *until* formula. Since we would like not to skip any vertex from $\sigma_c(t)$ we insist that the set E of minimal letters depending on c remains constant along the until move. This requirement does not ensure that no vertex from $\sigma_c(t)$ will be skipped but it is powerful enough for our purposes.

We start by defining some abbreviations. For $A \subseteq \Sigma$ we define the macro $(\min = A) \in \mathrm{GlobTL}_\Sigma$ by $\bigwedge_{a \in A} a \wedge \bigwedge_{a \notin A} \neg a$ so that for all $t \in \mathbb{R}$ we have $t \models_g (\min = A)$ if and only if $\min(t) = A$. We will also use macros $(\min \cap D(a) = A)$ for $a \in \Sigma$ and $A \subseteq \Sigma$ with the obvious meanings and definitions. Note that all these macros are robust. For a formula $\alpha \in \mathrm{GlobTL}_\Sigma$ and a subalphabet $E \subseteq \Sigma$, we define the global formula

$$\mathrm{shift}_{c,E}(\alpha) = (\min \cap D(c) = E)\ \mathsf{U}\ (\alpha \wedge (\min \cap D(c) = E)).$$

Lemma 7 *For all $t \in \mathbb{R}$, we have $t \models_g \mathrm{shift}_{c,E}(\alpha)$ if and only if $t = r_1 r_2 s$ with $r_1 r_2$ finite, $s \models_g \alpha$, $\mathrm{alph}(r_1) \subseteq E$, r_2 independent of $E \cup \{c\}$ and $E = \min(s) \cap D(c)$.*

Moreover, $\mathrm{shift}_{c,E}(\alpha)$ is robust if the following two conditions hold: α is robust and $s \models_g \alpha$ implies $rs \models_g \alpha$ for $r, s \in \mathbb{R}$ with r independent of s and c.



Proof. Assume first that $t \models_g \mathrm{shift}_{c,E}(\alpha)$ then we write $t = rs$ with r finite, $s \models_g \alpha$ and $\min(r''s) \cap D(c) = E$ for all $r'r'' = r$. Using $r'' = 1$ we get $E = \min(s) \cap D(c)$. There is a unique factorization $r = r_1 r_2$ with $\mathrm{alph}(r_1) \subseteq E$ and $\min(r_2) \cap E = \emptyset$. It remains to show that r_2 is independent of E and c. Assume that $r_2 = r_2' b r_2''$ with r_2' independent of $E \cup \{c\}$ and $r_2' b$ prime. We first show that $b \notin E$. If $r_2' = 1$ then $b \in \min(r_2)$ hence $b \notin E$. Otherwise, let $a \in \max(r_2')$. Since $(a, b) \in D$, we have $b \notin \min(abr_2''s)$ and $E \subseteq \min(abr_2''s)$, hence again $b \notin E$. Now, since $b \notin E = \min(br_2''s) \cap D(c)$, we deduce that b is independent of $E \cup \{c\}$.

Conversely, let $t = r_1 r_2 s$ with $r_1 r_2$ finite, $s \models_g \alpha$, $\mathrm{alph}(r_1) \subseteq E$, r_2 independent of $E \cup \{c\}$ and $E = \min(s) \cap D(c)$. We have to show that if $r'r'' = r_1 r_2$ then $E = \min(r''s) \cap D(c)$. By Levi's lemma, $r'r'' = r_1 r_2$ implies $r' = r_1' r_2'$, $r'' = r_1'' r_2''$, $r_1 = r_1' r_1''$ and $r_2 = r_2' r_2''$. Since $E \subseteq \min(s)$, the letters in E are pairwise independent and using $\mathrm{alph}(r_1) \subseteq E$, we get $E = \min(r_1''s) \cap D(c)$. Using in addition r_2 independent of $E \cup \{c\}$ we obtain $E = \min(r''s) \cap D(c)$. This proves the first part of the claim.

Assume now that α is robust and $s \models_g \alpha$ implies $rs \models_g \alpha$ for $r, s \in \mathbb{R}$ with r independent of s and c. Let $t \in \mathbb{R}$ be such that $t \models_g' \mathrm{shift}_{c,E}(\alpha)$ and write $t = rs$ with $s \models_g' \alpha$ and $\min(r''s) \cap D(c) = E$ for all $r'r'' = r$. Since α is robust, we get $s \models_g \alpha$. Now, write $r = r'r''$ with r' finite and $\mathrm{alph}(r'') = \mathrm{alphinf}(r)$. Then, r'' is independent of s. Now, if $b \in \mathrm{alph}(r'')$ then there is a suffix br''' of r and we obtain $\{b\} \cap D(c) \subseteq \min(br'''s) \cap D(c) = E$. Since b is independent of s and $E \subseteq \min(s)$, we deduce that $b \notin D(c)$. Therefore, $r''s \models_g \alpha$ and obtain easily $t = r'(r''s) \models_g \mathrm{shift}_{c,E}(\alpha)$. $\square$

Now, for $a \in \Sigma$, we let

$$(a \in \mu_c) = \bigvee_{E \,|\, a \notin E} \mathrm{shift}_{c,E}(a)$$

where the disjunction ranges over all subsets $E \subseteq \Sigma$ such that $a \notin E$.

Lemma 8 *For all $t \in \mathbb{R}$, we have $t \models_g (a \in \mu_c)$ if and only if $a \in \mathrm{alph}(\mu_c(t))$. Moreover, the formula $(a \in \mu_c)$ is robust.*

Proof. Let $t \in \mathbb{R}$ with $a \in \mathrm{alph}(\mu_c(t))$ and let $E = \min(t) \cap D(c) = \min(\sigma_c(t))$. We have $\mu_c(t)$ independent of $E \cup \{c\}$. In particular, $a \notin E$. We can write $\mu_c(t) = r_2 r'$ with r_2 finite and $a \in \min(r')$. Then, with $s = r'\sigma_c(t)$ we have $a \in \min(s)$ and $\min(s) \cap D(c) = E$. Hence, $t = r_2 s \models_g \mathrm{shift}_{c,E}(a)$ by Lemma 7 with $r_1 = 1$.

Conversely, assume that $t \models_g \mathrm{shift}_{c,E}(a)$ for some E with $a \notin E$. Let $t = r_1 r_2 s$ be the factorization given by Lemma 7. From $E = \min(s) \cap D(c)$, $a \in \min(s)$ and $a \notin E$, we deduce that a is independent of $E \cup \{c\}$. Using in addition that r_2 is independent of $E \cup \{c\}$ and $\mathrm{alph}(r_1) \subseteq E$ we infer that the trace $r_2 a$ is a prefix of $\mu_c(t)$ and we get $a \in \mathrm{alph}(\mu_c(t))$ as desired.

Finally, $\mathrm{shift}_{c,E}(a)$ is robust since $\alpha = a$ satisfies the additional requirement of Lemma 7 for robustness. $\square$

We deal now with $(\mathsf{X}_a \leq \mathsf{X}_b)$. If $a = b \in D(c)$ then we let $\overline{(\mathsf{X}_a \leq \mathsf{X}_a)}^c = \mathsf{F}\, a$ and if $a \neq b$ or if $a = b \notin D(c)$ then we let

$$\overline{(\mathsf{X}_a \leq \mathsf{X}_b)}^c = \bigvee_{E \mid b \notin E} \mathrm{shift}_{c,E}(\zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c))$$

where $\zeta = (\neg b)\, \mathsf{U}\, (a \wedge (b \notin \mu_a) \wedge \mathsf{F}\, b)$.

Lemma 9 *The formula* $\overline{(\mathsf{X}_a \leq \mathsf{X}_b)}^c$ *satisfies the requirement of Proposition 5.*

Proof. In all this proof, we let $x = \min_c(ct)$.

If $a = b \in D(c)$ then $a \notin \mathrm{alph}(\mu_c(t))$, hence $a \in \mathrm{alph}(t)$ if and only if $ct, \min_c(ct) \models (\mathsf{X}_a \leq \mathsf{X}_a)$. Moreover, the formula $\mathsf{F}\, a$ is robust.

Next, we consider the special case where $a, b \notin \mathrm{alph}(\mu_c(t))$. We show that for all $t \in \mathbb{R}$ such that $a, b \notin \mathrm{alph}(\mu_c(t))$, we have $t \models_g \zeta$ if and only if $ct, x \models (\mathsf{X}_a \leq \mathsf{X}_b)$.

Assume first that $a, b \notin \mathrm{alph}(\mu_c(t))$ and $t \models_g \zeta$. We can write $t = rs$ with $a \in \min(s)$ and $b \in \mathrm{alph}(s) \setminus \mathrm{alph}(\mu_a(s))$ and $b \notin \mathrm{alph}(r)$. We have $a, b \in \mathrm{alph}(t) \setminus \mathrm{alph}(\mu_c(t))$. We deduce that x_a and x_b exist. Let $y \in \min(s)$ with $\lambda(y) = a$. Since $a \notin \mathrm{alph}(\mu_c(t))$ we deduce that $x < y$ and therefore $x_a \leq y$. Now, $b \notin \mathrm{alph}(r) \cup \mathrm{alph}(\mu_a(s))$, hence x_b is in $\sigma_a(s)$ and we get $y = \min(\sigma_a(s)) \leq x_b$.

Conversely, note that if $a, b \notin \mathrm{alph}(\mu_c(t))$ then x_a and x_b are the first vertices labelled a and b respectively in the trace t. So assume that $a, b \notin \mathrm{alph}(\mu_c(t))$ and $x_a \leq x_b$ exist. We can write $t = rs$ with ra prime, $a \notin \mathrm{alph}(r)$ and $a \in \min(s)$. Since $a \notin \mathrm{alph}(\mu_c(t))$, we deduce that $x_a \in \min(s)$ and therefore $\{x_a\} = \min(\sigma_a(s))$. From $x_a \leq x_b$ we deduce that x_b is in $\sigma_a(s)$ and therefore $b \in \mathrm{alph}(s)$. Also, $b \notin \mathrm{alph}(\mu_a(s))$ since x_b is the first vertex labelled b in t. We have thus shown $s \models_g a \wedge (b \notin \mu_a) \wedge \mathsf{F}\, b$. Finally, let $r = r'r''$ with $r'' \neq 1$. We have $x_a \in r''s$ and the vertex x_a is not minimal in $r''s$ since ra is prime and $r'' \neq 1$. Then $b \notin \min(r''s)$ since $x_a \leq x_b$ and x_b is the first vertex of t which is labeled b. Therefore, $t \models_g \zeta$ as desired.

We show now that ζ is robust. Let $t \in \mathbb{R}$ with $t \models'_g \zeta$ and write $t = rs$ the associated factorization. Since $a \in \min(s)$, we can write $r = r_1 r_2$ with r_1 finite and r_2 independent of a. We have $\mu_a(r_2 s) = r_2 \mu_a(s)$ and $b \notin \mathrm{alph}(r)$, hence $b \in \mathrm{alph}(r_2 s) \setminus \mathrm{alph}(\mu_a(r_2 s))$. Also $a \in \min(r_2 s)$ and we deduce easily that $t = r_1(r_2 s) \models_g \zeta$.

For the general case, note that if $a \neq b$ or $a = b \notin D(c)$ then $ct, x \models (\mathsf{X}_a \leq \mathsf{X}_b)$ implies $b \notin \min(t) \cap D(c)$. Hence, it is enough to show that for all $t \in \mathbb{R}$ and $E \subseteq \Sigma$ with $b \notin E$, we have $E = \min(t) \cap D(c)$ and $ct, x \models (\mathsf{X}_a \leq \mathsf{X}_b)$ if and only if $t \models_g \mathrm{shift}_{c,E}(\zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c))$.

Let $t \in \mathbb{R}$ and $E \subseteq \Sigma$ with $b \notin E$. Assume that $E = \min(t) \cap D(c)$ and that x_a, x_b exist and $x_a \leq x_b$. We can write $\mu_c(t) = r_2 r'$ with r_2 finite and $\mathrm{alph}(r') = \mathrm{alphinf}(\mu_c(t))$. Since x_a, x_b exist, we have $a, b \in \mathrm{alph}(\sigma_c(t))$. Using $\sigma_c(t)$ independent of $\mathrm{alphinf}(\mu_c(t))$ we deduce that $a, b \notin \mathrm{alph}(r')$. With $s = r'\sigma_c(t)$, we have $\mu_c(s) = r'$ and $\sigma_c(s) = \sigma_c(t)$. Hence, $x = \min_c(cs)$ and $cs, x \models (\mathsf{X}_a \leq \mathsf{X}_b)$. Moreover, $a, b \notin \mathrm{alph}(\mu_c(s))$ and we deduce that $s \models_g \zeta$ from the special case above. Hence, we have shown $s \models_g \zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c)$. Now, we have $\mu_c(t)$ (hence also r_2) independent of $E \cup \{c\}$. Also, $E = \min(t) \cap D(c) = \min(\sigma_c(t)) = \min(\sigma_c(s)) = \min(s) \cap D(c)$. Using Lemma 7 with $r_1 = 1$, we obtain as desired $t = r_2 s \models_g \mathrm{shift}_{c,E}(\zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c))$.

Conversely, let $t \in \mathbb{R}$ be such that $t \models_g \mathrm{shift}_{c,E}(\zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c))$ for some E with $b \notin E$. Let $t = r_1 r_2 s$ be the factorization given by Lemma 7. Note that $E = \min(t) \cap D(c)$. Let $y = \min_c(cs)$. Since $a, b \notin \mathrm{alph}(\mu_c(s))$, we deduce from the special case above that y_a, y_b exist and $y_a \leq y_b$. We have $\sigma_c(t) = r_1 \sigma_c(s)$, hence x_a, x_b exist and $x_a \leq y_a$. Using $b \notin E$ and $\mathrm{alph}(r_1) \subseteq E$, we obtain $x_b = y_b$ and therefore, $x_a \leq x_b$.

Finally, the formula $\alpha = \zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c)$ is robust. Let $r, s \in \mathbb{R}$ with $s \models_g \alpha$ and r independent of s and c. We have $\mu_c(rs) = r\mu_c(s)$ and $\sigma_c(rs) = \sigma_c(s)$. Since $s \models_g \zeta$ we have $a, b \in \mathrm{alph}(s)$ and using r independent of s and $a, b \notin \mathrm{alph}(\mu_c(s))$ we deduce that $a, b \notin \mathrm{alph}(\mu_c(rs))$. Now, using the special case we have $s \models_g \zeta$ if and only if $\sigma_c(s) \models_g \zeta$ and $rs \models_g \zeta$ if and only if $\sigma_c(rs) \models_g \zeta$. Since $\sigma_c(s) = \sigma_c(rs)$ we deduce that $rs \models_g \zeta$. Therefore, α satisfies the additional requirement of Lemma 7 for robustness and we deduce that $\mathrm{shift}_{c,E}(\zeta \wedge (a \notin \mu_c) \wedge (b \notin \mu_c))$ is robust. $\square$

It remains to deal with $\varphi \,\mathsf{XU}_a\, \psi$. We define

$$\overline{\varphi \,\mathsf{XU}_a\, \psi}^c = (\xi \wedge (a \notin \mu_c)) \vee \bigvee_{E \mid a \notin E} \mathrm{shift}_{c,E}(\xi \wedge (a \notin \mu_c))$$

where $\xi = (\neg a \vee \langle a \rangle \overline{\varphi}^a) \mathsf{U} \langle a \rangle \overline{\psi}^a$.

Lemma 10 *The formula* $\overline{\varphi \mathsf{XU}_a \psi}^c$ *satisfies the requirement of Proposition 5.*

Proof. We first show that for all $t \in \mathbb{R}$ such that $a \notin \mathrm{alph}(\mu_c(t))$, we have $t \models_g \xi$ if and only if $ct, x \models \varphi \mathsf{XU}_a \psi$ where $x = \min_c(ct)$ is the minimal vertex of ct which is labelled c.

Assume first that $a \notin \mathrm{alph}(\mu_c(t))$ and $t \models_g \xi$. We can write $t = ras$ with $s \models_g \overline{\psi}^a$ and for all $r'r'' = r$ with $r'' \neq 1$, $r''as \models_g (\neg a \vee \langle a \rangle \overline{\varphi}^a)$. Let $z = \min_a(as)$ be the minimal vertex of as which is labelled a. By induction, we get $as, z \models \psi$. Since $a \notin \mathrm{alph}(\mu_c(t))$, we have $z \in \sigma_c(t)$ and $x < z$. Therefore $ct, z \models \psi$. Now, let $x < y < z$ with $\lambda(y) = a$. Then $y \in r$ and we have a factorization $r = r'ar''$ with $y = \min_a(ar'') = \min_a(ar''as)$. Since $ar'' \neq 1$ and $ar''as \models_g a$, we get $r''as \models_g \overline{\varphi}^a$. By induction we obtain $ar''as, y \models \varphi$ and therefore $ct, y \models \varphi$. We have thus shown that $ct, x \models \varphi \mathsf{XU}_a \psi$.

Conversely, assume that $a \notin \mathrm{alph}(\mu_c(t))$ and $ct, x \models \varphi \mathsf{XU}_a \psi$. We have to show that $t \models_g \xi$. Let $z > x$ with $\lambda(z) = a$ and $ct, z \models \psi$ and for all $x < y < z$ with $\lambda(y) = a$ we have $ct, y \models \varphi$. We can write $t = ras$ with ra prime and z being the maximal vertex of ra. Since $ct, z \models \psi$ we also have $as, z \models \psi$ and by induction we get $s \models_g \overline{\psi}^a$. Now, let $r'r'' = r$ with $r'' \neq 1$. By definition of r, z is not minimal in $r''as$. Assume that $a \in \min(r''as)$ and let $y = \min_a(r''as)$. We have $x < y < z$ since $a \notin \mathrm{alph}(\mu_c(t))$. Therefore $ct, y \models \varphi$ and also $r''as, y \models \varphi$. By induction, with $r'' = ar'''$, we obtain $r'''as \models_g \overline{\varphi}^a$ and $r''as \models_g \langle a \rangle \overline{\varphi}^a$. Therefore, $t \models_g \xi$.

Next, we show that ξ is robust. Assume that $t \models'_g \xi$ and write $t = ras$ with $s \models'_g \overline{\psi}^a$ and $r''as \models'_g \neg a \vee \langle a \rangle \overline{\varphi}^a$ for all $r'r'' = r$ with $r \neq 1$. Since $\overline{\psi}^a$ is robust we also have $s \models_g \overline{\psi}^a$. We can write $r = r_1 r_2$ with r_1 finite and r_2 independent of a. We have $s \models_g \overline{\psi}^a$ if and only if $as, \min_a(as) \models \psi$ if and only if $a\sigma_a(s), \min(a\sigma_a(s)) \models \psi$. Since $\sigma_a(s) = \sigma_a(r_2 s)$ and $s \models_g \overline{\psi}^a$, we deduce $r_2 s \models_g \overline{\psi}^a$. Also, for all $r'r'' = r_1$ with $r'' \neq 1$ we have $r''ar_2 s \models'_g \neg a \vee \langle a \rangle \overline{\varphi}^a$, hence also $r''ar_2 s \models_g \neg a \vee \langle a \rangle \overline{\varphi}^a$ since $\langle a \rangle \overline{\varphi}^a$ is robust. Therefore, $t \models_g \xi$.

Note that if $a \in \mathrm{alph}(\mu_c(t))$ then $a \notin E = \min(t) \cap D(c)$. Hence, it remains to show that for all $t \in \mathbb{R}$, $x = \min_c(ct)$ and $E \subseteq \Sigma$ with $a \notin E$, we have $E = \min(t) \cap D(c)$ and $ct, x \models \varphi \mathsf{XU}_a \psi$ if and only if $t \models_g \mathrm{shift}_{c,E}(\xi \wedge (a \notin \mu_c))$.

Let $t \in \mathbb{R}$ and $E \subseteq \Sigma$ with $a \notin E$. Assume that $E = \min(t) \cap D(c)$ and that $ct, x \models \varphi \mathsf{XU}_a \psi$. We can write $\mu_c(t) = r_2 r'$ with r_2 finite and $\mathrm{alph}(r') = \mathrm{alphinf}(\mu_c(t))$. We have $a \in \mathrm{alph}(\sigma_c(t))$. Using $\sigma_c(t)$ independent of $\mathrm{alphinf}(\mu_c(t))$ we deduce that $a \notin \mathrm{alph}(r')$. With $s = r'\sigma_c(t)$, we have $\mu_c(s) = r'$ and $\sigma_c(s) = \sigma_c(t)$. Hence $a \notin \mathrm{alph}(\mu_c(s))$ and $cs, x \models \varphi \mathsf{XU}_a \psi$.

We deduce that $s \models_g \xi$ from the special case above. We have thus shown $s \models_g \xi \wedge (a \notin \mu_c)$. Now, we have $\mu_c(t)$ (hence also r_2) independent of $E \cup \{c\}$. Also, $E = \min(t) \cap D(c) = \min(\sigma_c(t)) = \min(\sigma_c(s)) = \min(s) \cap D(c)$. Using Lemma 7 with $r_1 = 1$, we obtain as desired $t = r_2 s \models_g \mathrm{shift}_{c,E}(\xi \wedge (a \notin \mu_c))$.

Conversely, let $t \in \mathbb{R}$ be such that $t \models_g \mathrm{shift}_{c,E}(\xi \wedge (a \notin \mu_c))$ for some E with $a \notin E$. Let $t = r_1 r_2 s$ be the factorization given by Lemma 7. Note that $E = \min(t) \cap D(c)$. Since $a \notin \mathrm{alph}(\mu_c(s))$, we deduce from the special case above that $cs, \min_c(cs) \models \varphi \,\mathsf{XU}_a\, \psi$. We have $\sigma_c(t) = r_1 \sigma_c(s)$. Using $a \notin E$ and $\mathrm{alph}(r_1) \subseteq E$, we obtain $ct, x \models \varphi \,\mathsf{XU}_a\, \psi$.

Finally, we can show exactly as in the proof of Lemma 9 that $\alpha = \xi \wedge (a \notin \mu_c)$ satisfies the additional requirement of Lemma 7 for robustness. Therefore, the formula $\mathrm{shift}_{c,E}(\xi \wedge (a \notin \mu_c))$ is robust. $\square$

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