Some Remarks about Stabilizers

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Abstract

We continue our study of stabilizers of infinite words over finite alphabets, began in [10]. Let $\mathbf{w}$ be an aperiodic infinite word over a finite alphabet, and let $\mathcal{S}tab(\mathbf{w})$ be its stabilizer. We show that $\mathcal{S}tab(\mathbf{w})$ can be partitioned into the monoid of morphisms that stabilize $\mathbf{w}$ by finite fixed points and the ideal of morphisms that stabilize $\mathbf{w}$ by iteration. We also settle a conjecture of [10] by showing that in some cases $\mathcal{S}tab(\mathbf{w})$ is infinitely generated. If the aforementioned ideal is nonempty, then it contains either polynomially growing morphisms or exponentially growing morphisms, but not both. Moreover, in the polynomial case, the degree of polynomial is fixed. We also show how to compute the polynomial degree from the dependency graph of a polynomially growing morphism.

Key words: Infinite words; Morphisms; Stabilizers

This paper is dedicated to Juhani Karhumäki for his 60th birthday.

1. Introduction

In this paper we continue the study of stabilizers of aperiodic infinite words, which we began in [10]. Let $\mathbf{w}$ be an infinite word over a finite alphabet Σ. The *stabilizer* of $\mathbf{w}$ is the monoid of morphisms $h : \Sigma^* \to \Sigma^*$ that satisfy $h(\mathbf{w}) = \mathbf{w}$.

The previous paper was concerned mainly with questions related to the minimal number of generators a given stabilizer has. The current paper is concerned with questions related to the algebraic structure of stabilizers, and

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the growth order of morphisms in a given stabilizer. After stating some notation in Section 2, we show in Section 3 that $\mathcal{S}tab(\mathbf{w})$ is a disjoint union of the monoid of morphisms that stabilize $\mathbf{w}$ by finite fixed points, and the ideal of morphisms that generate $\mathbf{w}$ by iteration. The aforementioned monoid is always finite and nonempty, while the ideal is either empty or infinite. Contrary to what was conjectured in [10], this ideal can be infinitely generated, as we also demonstrate in Section 3.

In Section 4, we consider the growth order of $\mathbf{w}$ under a morphism h that generates it by iteration. We show that a given stabilizer can contain either morphisms under which $\mathbf{w}$ grows polynomially or morphisms under which $\mathbf{w}$ grows exponentially, but not both. Moreover, if the growth is polynomial, then the degree of polynomial is fixed for all stabilizer elements. This result enables us to extend Durand's generalization of Cobham's theorem [2, 3] to a wider family of morphisms.

2. Notation

For a finite alphabet Σ, the sets of finite words, nonempty finite words, and right-infinite words over Σ are denoted by Σ^*, Σ^+, and Σ^ω, respectively. The empty word is denoted by ε. The set of letters occurring in a word w is denoted by $\mathrm{alph}(w)$. A *factor* of a word $w \in \Sigma^* \cup \Sigma^\omega$ is a word $u \in \Sigma^*$ such that $w = xuy$ for some $x \in \Sigma^*$ and $y \in \Sigma^* \cup \Sigma^\omega$. The set of factors occurring in a word w is denoted by $\mathrm{Fact}(w)$. An infinite word $\mathbf{w}$ is *ultimately periodic* if $\mathbf{w} = xy^\omega$ for some $x \in \Sigma^*$ and $y \in \Sigma^+$, where $y^\omega = yyy\cdots$; otherwise, $\mathbf{w}$ is *aperiodic*. The length of a finite word u is denoted by $|u|$. The number of times a letter a occurs in a finite word u is denoted by $|u|_a$. The identity morphism is denoted by Id. The *width* of a morphism $h : \Sigma^* \to \Sigma^*$, denoted by $\|h\|$, is defined by

$$\|h\| = \max\{|h(a)| : a \in \Sigma\}.$$

The *stabilizer* of an infinite word $\mathbf{w}$, denoted by $\mathcal{S}tab(\mathbf{w})$, is the monoid of morphisms defined over $\Sigma = \mathrm{alph}(\mathbf{w})$ that fix $\mathbf{w}$:

$$\mathcal{S}tab(\mathbf{w}) = \{h : \Sigma^* \to \Sigma^* : h(\mathbf{w}) = \mathbf{w}\}.$$

3. Algebraic aspects of stabilizers

Let Σ be a finite alphabet and let $h : \Sigma^* \to \Sigma^*$ be a morphism. A letter $a \in \Sigma$ is said to be *mortal* under h if there exists some $t \geq 1$ such that $h^t(a) = \varepsilon$. The set of all mortal letters associated with h is denoted by M_h. A word is mortal if it belongs to M_h^*; otherwise it is *immortal*. A letter $a \in \Sigma$ is said to be *monorecursive* under h if $h(a) \in M_h^* a M_h^*$. Note that a monorecursive letter is immortal. Let $F_h = \{h^{|\Sigma|}(a) : a \text{ is monorecursive}\}$. It is well known and easy to see that the set of finite fixed points of h is equal to F_h^* ([6]; see also [1, Theorem 7.2.3]).

As an introduction to the methods in the field we recall some basic observations. Let $\mathbf{w} \in \Sigma^\omega$, and let $h \in \mathcal{S}tab(\mathbf{w})$. Then $\mathbf{w}$ can be factorized uniquely as $\mathbf{w} = u_0 a_0 u_1 a_1 \cdots$, where u_i are mortal words and a_i are immortal letters for all $i \geq 0$. since $\mathbf{w} = h(\mathbf{w})$ and $\mathbf{w}$ is infinite, there are infinitely many a_i's. If a_i is monorecursive for all $i \geq 0$, then $\mathbf{w}$ is an infinite product of finite fixed points of h. Otherwise, let $a = a_j$ be the first non-monorecursive letter in $\mathbf{w}$, and let $u = u_0 a_0 \cdots u_j$. Since $h(\mathbf{w}) = \mathbf{w}$ and $h(u)$ contains only mortal and monorecursive letters, necessarily $h(a) = yax$ for some $y, x \in \Sigma^*$. Since $h(a_i)$ contains a_i for all $0 \leq i < j$, the word y has to be mortal. Therefore, the word x has to be immortal, else we would get that a is monorecursive. This implies that $\mathbf{w}$ can be generated by iterating h on ua: $\mathbf{w} = h^\omega(ua) = uaxh(x)h^2(x) \cdots = vh^\omega(a)$, where $u = vh^{|\Sigma|-1}(y) \cdots h(y)y$. We recover the following classical theorem:

Theorem 1 (Head and Lando [7]). *Let $h : \Sigma^* \to \Sigma^*$, and let $\mathbf{w} \in \Sigma^\omega$. Then $\mathbf{w}$ is a fixed point of h if and only if exactly one of the following two conditions holds:*

1. $\mathbf{w} \in F_h^\omega$;
2. $\mathbf{w} = vh^\omega(a)$, *where $v \in F_h^*$ and $h(a) = yax$, with y mortal and x immortal.*

See also Allouche and Shallit, [1, Section 7.2–7.3].

Definition1. Let $\mathbf{w} \in \Sigma^\omega$. The *finite stabilizer* of $\mathbf{w}$, denoted by $\mathcal{FS}tab(\mathbf{w})$, is the set of morphisms that stabilize $\mathbf{w}$ by finite fixed points; the *morphic stabilizer* of $\mathbf{w}$, denoted by $\mathcal{MS}tab(\mathbf{w})$, is the set of morphisms that stabilize $\mathbf{w}$ by iteration. By the above discussion, $\mathcal{S}tab(\mathbf{w})$ is a disjoint union of $\mathcal{FS}tab(\mathbf{w})$ and $\mathcal{MS}tab(\mathbf{w})$.

Proposition 2. *Let $\mathbf{w} \in \Sigma^\omega$, let $\sigma = |\Sigma|$, and let $h \in \mathcal{S}tab(\mathbf{w})$. Then*

$$\max\{|u| : u \in Fact(\mathbf{w}) \cap M_h^*\} < 2\|h^\sigma\|.$$

Proof. Let $\mathbf{w} = u_0 a_0 u_1 a_1 \cdots$, where $u_i \in M_h^*$ and $a_i \in \Sigma$ are immortal for all $i \geq 0$. Then $h^\sigma(u_i) = \varepsilon$ for all $i \geq 0$. Since $\mathbf{w} = h^\sigma(\mathbf{w}) = h^\sigma(a_0 a_1 a_2 \cdots)$, and since $h^\sigma(a_i)$ contains at least one immortal letter for all $i \geq 0$, the inequality follows. $\square$

Proposition 3. *Let $\mathbf{w} \in \Sigma^\omega$.*

1. *Let $h \in \mathcal{MS}tab(\mathbf{w})$. Then $h^n \neq h^m$ for all $n \neq m$. In particular, $\mathcal{MS}tab(\mathbf{w})$ is either empty or infinite.*
2. *$\mathcal{FS}tab(\mathbf{w})$ is nonempty and finite.*

Proof. Suppose $\mathcal{MS}tab(\mathbf{w})$ is not empty, and let $h \in \mathcal{MS}tab(\mathbf{w})$. By assumption, $\mathbf{w} = vh^\omega(a)$, where $v \in F_h^*$, and $a \in \Sigma$ satisfies $h(a) = yax$, with y mortal and x immortal. Therefore, $h^n(a) \neq h^m(a)$ for all $n \neq m$, and so $\mathcal{MS}tab(\mathbf{w})$ contains the infinite set $\{h^n : n > 0\}$.

Now consider $\mathcal{FS}tab(\mathbf{w})$. Since $\mathrm{Id} \in \mathcal{FS}tab(\mathbf{w})$, $\mathcal{FS}tab(\mathbf{w})$ is nonempty. Also, if $h \in \mathcal{FS}tab(\mathbf{w})$, then all letters are either mortal or monorecursive under

h. Fix a set $A \subseteq \Sigma$, and let $\mathcal{FS}tab_A(\mathbf{w})$ be the set of $\mathcal{FS}tab(\mathbf{w})$ elements h that satisfy $M_h = A$. It is enough to show that $\mathcal{FS}tab_A(\mathbf{w})$ is finite.

Suppose $\mathcal{FS}tab_A(\mathbf{w})$ is nonempty. Then $\mathbf{w} = u_0a_0u_1a_1\cdots$, where $u_i \in M_h^*$ and $a_i \in \Sigma$ is monorecursive under h for all $h \in \mathcal{FS}tab_A(\mathbf{w})$ and for all $i \geq 0$. Also, by Proposition 2, $\max\{|u_i| : i \geq 0\}$ is bounded by a constant, say K. Let $h \in \mathcal{FS}tab_A(\mathbf{w})$. Then $|h(a_i)| \leq |u_i| + 1 + |u_{i+1}| \leq 2K + 1$ for all $i \geq 0$, or we would get that a_i is not monorecursive. Similarly, $|h(b)| \leq |u_i| \leq K$ for all $i \geq 0$ and for all letters b occurring in u_i, or we would get that b is immortal. This implies there can be only finitely many morphisms $h \in \mathcal{FS}tab(\mathbf{w})$ satisfying $M_h = A$, that is, $\mathcal{FS}tab_A(\mathbf{w})$ is finite. $\square$

Proposition 4. *Let $\mathbf{w} \in \Sigma^\omega$, let $\sigma = |\Sigma|$, and let $h \in \mathcal{FS}tab(\mathbf{w})$. Then for infinitely many prefixes p of $\mathbf{w}$ we have $h(p) = p$, and for all prefixes p,*

$$\big|\,|h(p)| - |p|\,\big| \leq \|h^\sigma\|.$$

Proof. Let $h \in \mathcal{FS}tab$. Then $\mathbf{w} = x_0x_1x_2\cdots$, where $x_i \in F_h$ for all $i \geq 0$, and so every prefix p of $\mathbf{w}$ has the form $p = x_0\cdots x_{k-1}y$, where y is a prefix of x_k. If $y = \varepsilon$ then $h(p) = p$. Otherwise,

$$|x_0\cdots x_{k-1}| = |h(x_0\cdots x_{k-1})| \leq |h(p)| \leq |h(x_0\cdots x_{k-1}x_k)| = |x_0\cdots x_{k-1}x_k|,$$

and so

$$\big|\,|h(p)| - |p|\,\big| \leq \max\{|x| : x \in F_h\} \leq \max\{|h^\sigma(a)| : a \in \Sigma\} = \|h^\sigma\|.$$

$\square$

Proposition 5. *Let $\mathbf{w} \in \Sigma^\omega$, and let $h \in \mathcal{MS}tab(\mathbf{w})$. Then $|h(p)| > |p|$ for almost all prefixes p of $\mathbf{w}$.*

Proof. Let $h \in \mathcal{MS}tab(\mathbf{w})$. Then there exists some $u \in F_h^*$ and a letter a, such that $h(a) = yax$ with y mortal and x immortal, and $\mathbf{w} = h^\omega(ua) = uaxh(x)h^2(x)\cdots$. For $i \geq 0$, let $p_i = h^i(ua) = uaxh(x)\cdots h^{i-1}(x)$. Let p be a prefix of $\mathbf{w}$ with $|p| \geq |p_0|$. Then $p = p_ip'$ for some $i \geq 0$, where p' is a prefix of $h^i(x)$. Therefore,

$$|h(p)| = |h(p_i)h(p')| = |p_ih^i(x)h(p')| > |p_ip'| = |p|.$$

$\square$

Corollary 6. *Let $\mathbf{w} \in \Sigma^\omega$ and let $h \in \mathcal{S}tab(\mathbf{w})$. Then $h \in \mathcal{MS}tab(\mathbf{w})$ if and only if $|h(p)| > |p|$ for almost all prefixes p of $\mathbf{w}$.*

Proposition 7. *Let $\mathbf{w} \in \Sigma^\omega$. Then $\mathcal{MS}tab(\mathbf{w})$ is a subsemigroup of $\mathcal{S}tab(\mathbf{w})$.*

Proof. Let $h, g \in \mathcal{MS}tab(\mathbf{w})$. Then there exists some $n > 0$ such that $|h(p)| > |p|$ and $|g(p)| > |p|$ for any prefix p of $\mathbf{w}$ with $|p| > n$. For such a prefix p, assume w.l.o.g. that $h(p) = px$ and $g(p) = pxy$, with $x \in \Sigma^+$ and $y \in \Sigma^*$. Then $|hg(p)| = |pxh(xy)| > |p|$ and $|gh(p)| = |pxyg(x)| > |p|$. We get that $|hg(p)|, |gh(p)| > |p|$ for all prefixes p with $|p| > n$, and so, by Corollary 6, $hg, gh \in \mathcal{MS}tab(\mathbf{w})$. $\square$

Proposition 8. *Let* $\mathbf{w} \in \Sigma^\omega$ *be aperiodic, and let* $h \in \mathcal{MS}tab(\mathbf{w})$. *Then* $\lim_{|p|\to\infty} \{|h(p)| - |p| :$ *p is a prefix of* $w\} = \infty$.

Proof. As in Proposition 5, let $\mathbf{w} = h^\omega(ua) = uaxh(x)h^2(x)\cdots$, and let $p_i = h^i(ua) = uaxh(x)\cdots h^{i-1}(x)$. Assume that there exists some constant $C > 0$ such that $|h^i(x)| = |h(p_i)| - |p_i| \leq C$ for infinitely many i's. Then there exist some integers $j \neq k$ such that $h^j(x) = h^k(x)$, and so we get that $\mathbf{w}$ is ultimately periodic, a contradiction. Hence, for all $C > 0$ we have $|h^i(x)| = |h(p_i)| - |p_i| > C$ for almost all i.

Now assume that there exists some constant $C > 0$ such that $|h(p)|-|p| \leq C$ for infinitely many prefixes p. Let p be such a prefix, let i be such that $|p_{i-1}| < |p| < |p_i|$, and assume that $|h^i(x)| > C'$, where $C' = C(\|h\| + 1)$ (by the above observation we can pick such p). Then $|h^2(p_{i-1})| = |p_{i+1}| \leq |h^2(p)|$, and so $|h^2(p)| - |p| \geq |h^i(x)| > C'$. But by Proposition 5, $h(p) = pv$ for some $v \in \Sigma^+$, and by assumption, $|v| = |h(p)| - |p| \leq C$. We get that $|h^2(p)| - |p| = |vh(v)| \leq C'$, a contradiction. Therefore, for all C we have $|h(p)| - |p| > C$ for almost all prefixes p. □

Definition2. Let $\mathbf{w} \in \Sigma^\omega$, and let $h \in \mathcal{S}tab(\mathbf{w})$. We say that h satisfies the *bounded prefix property* if there exists a constant $C > 0$ such that $\big|\,|h(p)| - |p|\,\big| \leq C$ for all prefixes p of $\mathbf{w}$.

Corollary 9. *Let* $\mathbf{w} \in \Sigma^\omega$ *be aperiodic, and let* $h \in \mathcal{S}tab(\mathbf{w})$. *Then* $h \in \mathcal{FS}tab(\mathbf{w})$ *if and only if* h *satisfies the bounded prefix property.*

Proposition 10. *Let* $\mathbf{w} \in \Sigma^\omega$ *be aperiodic. Then* $\mathcal{FS}tab(\mathbf{w})$ *is a submonoid of* $\mathcal{S}tab(\mathbf{w})$.

Proof. Let $h, g \in \mathcal{FS}tab(\mathbf{w})$. Then h, g satisfy the bounded prefix property, with constants C_h, C_g, respectively. Let $C = \max\{C_h, C_g\}$. Then

$$\big|\,|hg(p)| - |p|\,\big| = \big|\,|h(g(p))| - |g(p)| + |g(p)| - |p|\,\big| \leq \big|\,|h(g(p))| - |g(p)|\,\big| + \big|\,|g(p)| - |p|\,\big| \leq 2C.$$

We get that hg satisfies the bounded prefix property with constant $2C$, and so $hg \in \mathcal{FS}tab(\mathbf{w})$ by Corollary 9. Since $\mathrm{Id} \in \mathcal{FS}tab(\mathbf{w})$, we get that $\mathcal{FS}tab(\mathbf{w})$ is a monoid. □

Example1. Proposition 10 does not hold if $\mathbf{w}$ is ultimately periodic. Let $\Sigma = \{0, 1, 2\}$ and let $\mathbf{w} = 0(12)^\omega$. Define $h, g : \Sigma^* \to \Sigma^*$ by

$$h = \{0 \to 01, 1 \to \varepsilon, 2 \to 21\}, \;\; g = \{0 \to 0, 1 \to 12, 2 \to \varepsilon\}.$$

Then $h, g \in \mathcal{FS}tab(\mathbf{w})$, but $gh = \{0 \to 012, 1 \to \varepsilon, 2 \to 12\} \in \mathcal{MS}tab(\mathbf{w})$.

A subsemigroup $\mathcal{A} \subseteq \mathcal{MS}tab(\mathbf{w})$ is a *right ideal* of $\mathcal{MS}tab(\mathbf{w})$ if $hg \in \mathcal{A}$ for all $h \in \mathcal{A}$ and $g \in \mathcal{MS}tab(\mathbf{w})$. Similarly, $\mathcal{A}$ is a *left ideal* if $gh \in \mathcal{A}$ for all $h \in \mathcal{A}$ and $g \in \mathcal{MS}tab(\mathbf{w})$. If $\mathcal{A}$ is both a left and a right ideal then it is an *ideal* of $\mathcal{MS}tab(\mathbf{w})$.

Proposition 11. *Let* $\mathbf{w} \in \Sigma^\omega$ *be aperiodic. Then* $\mathcal{MS}tab(\mathbf{w})$ *is an ideal of* $\mathcal{S}tab(\mathbf{w})$.

Proof. By Proposition 7, $\mathcal{MS}tab(\mathbf{w})$ is closed under composition. We show that if $h \in \mathcal{MS}tab(\mathbf{w})$ and $g \in \mathcal{FS}tab(\mathbf{w})$, then both hg and gh do not satisfy the bounded prefix property.

As in Proposition 8, let $\mathbf{w} = h^\omega(ua) = uaxh(x)h^2(x)\cdots$, and let $p_i = h^i(ua) = uaxh(x)\cdots h^{i-1}(x)$. Since $\mathbf{w}$ is aperiodic, $\lim_{i\to\infty} |h^i(x)| = \infty$. Suppose gh satisfies the bounded prefix property with constant C. Then for all $i \geq 0$,

$$C \geq \big|\, |gh(p_i)| - |p_i| \,\big| = \big|\, |g(p_{i+1})| - |p_i| \,\big| = \big|\, |g(p_i h^i(x))| - |p_i| \,\big| = \big|\, |g(p_i)| - |p_i| + |g(h^i(x))| \,\big|.$$

Since $g \in \mathcal{FS}tab(\mathbf{w})$, we get by Proposition 4 that $|g(p_i)| > |p_i| - C'$, where $C' = \|g^{|\Sigma|}\|$. Since the factors of $\mathbf{w}$ that are mortal under g are of bounded length, and $\lim_{i\to\infty} |h^i(x)| = \infty$, we can choose some i such that $h^i(x)$ contains more than $C + C'$ immortal letters under g. This implies that $|g(p_i)| - |p_i| + |g(h^i(x))| > C$, a contradiction.

Now suppose that hg satisfies the bounded prefix property. Then for all $i \geq 0$, $\big|\, |hg(p_i)| - |p_i| \,\big| \leq C$. Since $|g(p_i)| > |p_i| - C'$, we get that $|h(g(p_i))| > |h(p_i)| - C''$, where $C'' = C'\|h\|$. Since $\lim_{i\to\infty}(|h(p_i)| - |p_i|) = \infty$, we can choose i such that $|h(p_i)| > |p_i| + C'' + C$, and so $|h(g(p_i))| - |p_i| > C$, a contradiction. □

Example2. Proposition 11 does not hold if $\mathbf{w}$ is ultimately periodic. Let $\Sigma = \{0, 1, 2, 3\}$ and let $\mathbf{w} = 01(23)^\omega$. Define $h, g : \Sigma^* \to \Sigma^*$ by

$$h = \{0 \to 01, 1 \to 2, 2 \to 3, 3 \to 2\}, \quad g = \{0 \to 01, 1 \to \varepsilon, 2 \to \varepsilon, 3 \to 23\}.$$

Then $h \in \mathcal{MS}tab(\mathbf{w})$ and $g \in \mathcal{FS}tab(\mathbf{w})$, but

$$gh = \{0 \to 01, 1 \to \varepsilon, 2 \to 23, 3 \to \varepsilon\} \in \mathcal{FS}tab(\mathbf{w}).$$

Corollary 12. *Let* $\mathbf{w} \in \Sigma^\omega$ *be aperiodic. Then*

1. $\mathcal{S}tab(\mathbf{w})$ *is finite if and only if* $\mathcal{MS}tab(\mathbf{w})$ *is empty;*
2. $\mathcal{S}tab(\mathbf{w})$ *is a finitely generated monoid if and only if* $\mathcal{MS}tab(\mathbf{w})$ *is a finitely generated semigroup.*

Proof. By Propositions 3, 10, 11, $\mathcal{FS}tab(\mathbf{w})$ is a finite monoid and $\mathcal{MS}tab(\mathbf{w})$ is an ideal, either empty or infinite. The result follows. □

In [10], it was conjectured that stabilizers of aperiodic infinite words are always finitely generated. This conjecture turns out to be false, as the following theorem illustrates.

Theorem 13. *There exists an aperiodic infinite word* $\mathbf{w}$ *over a ternary alphabet such that* $\mathcal{S}tab(\mathbf{w})$ *is infinitely generated.*

PROOF. Let $\Sigma = \{a, b, c\}$. Define an infinite word $\mathbf{w} = w_0 w_1 w_2 \cdots \in \Sigma^\omega$ by

$$w_i = \begin{cases} a, & \text{if} \quad i = 0, \\ b, & \text{if} \quad i = 2^j \text{ for some } j \geq 0, \\ c, & \quad\quad \text{otherwise.} \end{cases}$$

Then $\mathbf{w}$ is aperiodic. Let $u_{-1} = ab$, and for $k \geq 0$, let

$$u_k = u_{k-1} c^{2^k - 1} b = abbcb \cdots bc^{2^k - 1} b.$$

Then every prefix p of $\mathbf{w}$ with $|p| \geq 2$ has the form $p = u_k c^m$ for some $k \geq -1$ and $0 \leq m < 2^{k+1}$. For such a pair (k, m), define the morphism $h_{k,m} : \Sigma^* \to \Sigma^*$ by

$$h_{k,m}(\alpha) = \begin{cases} u_k c^m, & \text{if} \quad \alpha = a, \\ c^{2^{k+1} - m - 1} b c^{2^{k+1} + m}, & \text{if} \quad \alpha = b, \\ c^{2^{k+2}}, & \text{if} \quad \alpha = c. \end{cases}$$

To prove that $\mathcal{S}tab(\mathbf{w})$ is infinitely generated, we first prove the following lemma:

Lemma 14. $\mathcal{S}tab(\mathbf{w}) = \{\mathrm{Id}\} \cup \{h_{k,m} : k \geq -1,\ 0 \leq m < 2^{k+1}\}$.

PROOF. First, we show that all nontrivial stabilizer elements have the form $h_{k,m}$ for some $k \geq -1$ and $0 \leq m < 2^{k+1}$. Let $f \in \mathcal{S}tab(\mathbf{w})$, $f \neq \mathrm{Id}$. The construction of $\mathbf{w}$ implies the following properties:

1. a occurs exactly once in $\mathbf{w}$;
2. both b and cc occur infinitely often in $\mathbf{w}$;
3. $bc^i b$ occurs in $\mathbf{w}$ if and only if $i = 2^k - 1$ for some $k \geq 0$, in which case it occurs exactly once.

These properties imply that f is nonerasing: if either $f(b) = \varepsilon$ or $f(c) = \varepsilon$ then $\mathbf{w}$ is ultimately periodic, a contradiction; if $f(a) = \varepsilon$ then necessarily $f(b) = ax$ for some $x \in \Sigma^*$, and so a occurs infinitely often in $\mathbf{w}$, a contradiction. For the same reason, both $f(b)$ and $f(c)$ do not contain the letter a.

Suppose $f(c)$ contains the letter b. Then $f(cc)$ contains a factor of the form $bc^n b$, and so $bc^n b$ occurs infinitely often in $\mathbf{w}$, a contradiction to property 3. Therefore, $f(c) = c^\ell$ for some $\ell > 0$. Since b occurs infinitely often in $\mathbf{w}$, this implies that $f(b)$ contains at least one b. Suppose that $f(b)$ contains more than one b. Then $f(b)$ contains a factor of the form $bc^n b$, and again we get that $bc^n b$ occurs infinitely often in $\mathbf{w}$. Therefore, $f(b)$ contains exactly one b. This implies that $|f(a)| \geq 2$: otherwise, since $\mathbf{w}$ begins with abb, necessarily $f = \mathrm{Id}$, a contradiction.

We conclude that f satisfies the following:

- $f(a) = u_k c^m$ for some $k \geq -1$ and $0 \leq m < 2^{k+1}$;
- $f(b) = c^i b c^j$ for some $i, j \geq 0$;
- $f(c) = c^\ell$ for some $\ell > 0$.

We now express i, j, ℓ in terms of k and m. For $f(b)$,

$$f(abb) = u_k c^m \cdot c^i b c^j \cdot c^i b c^j = u_k \cdot c^{2^{k+1}-1} \cdot b \cdot c^{2^{k+2}-1} \cdot b \cdot c^j.$$

Therefore,

$$\begin{array}{rcl} m+i & = & 2^{k+1}-1, \\ i+j & = & 2^{k+2}-1, \end{array}$$

and so $i = 2^{k+1} - m - 1$ and $j = 2^{k+1} + m$. For $f(c)$,

$$f(abbcb) = f(abb) \cdot f(c) \cdot f(b) = u_{k+2} c^{2^{k+1}+m} \cdot c^{\ell} \cdot c^{2^{k+1}-m-1} b c^{2^{k+1}+m} = \\ u_{k+2} c^{2^{k+3}-1} \cdot b c^{2^{k+1}+m}.$$

Therefore, $2^{k+1} + m + \ell + 2^{k+1} - m - 1 = 2^{k+3} - 1$, and $\ell = 2^{k+2}$. We get that $f = h_{k,m}$. This completes the proof of the first direction of Lemma 14.

For the other direction, we need to show that $h_{k,m} \in \mathcal{S}tab(\mathbf{w})$ for all appropriate pairs (k, m). Since $\mathbf{w} = \lim_{j\to\infty} u_j$, it is enough to show that $h_{k,m}(u_j)$ is a prefix of $\mathbf{w}$ for all pairs (k, m) and for all $j \geq -1$. We prove by induction on j that

$$h_{k,m}(u_j) = u_{k+j+2} c^{2^{k+1}+m}. \tag{1}$$

Let $f = h_{k,m}$. By definition,

$$f(u_{-1}) = f(ab) = u_k c^m \cdot c^{2^{k+1}-m-1} b c^{2^{k+1}+m} = \\ u_k c^{2^{k+1}-1} b c^{2^{k+1}+m} = u_{k+1} c^{2^{k+1}+m}.$$

Now assume that $f(u_{j-1}) = u_{k+j+1} c^{2^{k+1}+m}$. Then

$$f(u_j) = f(u_{j-1} \cdot c^{2^j-1} \cdot b) = u_{k+j+1} c^{2^{k+1}+m} \cdot c^{2^{k+2}(2^j-1)} \cdot c^{2^{k+1}-m-1} b c^{2^{k+1}+m} = \\ u_{k+j+1} c^{2^{k+j+2}-1} b c^{2^{k+1}+m} = u_{k+j+2} c^{2^{k+1}+m}.$$

This completes the proof of Lemma 14. □

We now continue to prove Theorem 13. Suppose that $\mathcal{S}tab(\mathbf{w})$ is finitely generated, and consider the set $\{h_{k,0} : k \geq -1\}$. Then there exists some $N \in \mathbb{N}$ such that for all $k > N$, $h_{k,0} = fg$ for some nontrivial $f, g \in \mathcal{S}tab(\mathbf{w})$. But by Lemma 14, there exists some integers i, j, m, n, with $i, j \geq -1$, $0 \leq m < 2^{i+1}$, and $0 \leq n < 2^{j+1}$, such that $f = h_{i,m}$ and $g = h_{j,n}$. Therefore (recall (1)),

$$fg(a) = h_{i,m}(u_j c^n) = u_{i+j+2} c^{2^{i+1}+m} \cdot c^{n 2^{i+2}} = u_{i+j+2} c^{(2n+1)2^{i+1}+m}.$$

By Lemma 14, this implies that $fg = h_{k',m'}$, where $k' = i + j + 2$ and $m' = (2n + 1)2^{i+1} + m$. Since $2n + 1 \geq 1$ and $i + 1 \geq 0$, we get that $m' > 0$, a contradiction: we assumed that $fg = h_{k,0}$. Therefore, $\mathcal{S}tab(\mathbf{w})$ is infinitely generated. □

A monoid $\mathcal{M}$ is called *aperiodic* if for all $m \in \mathcal{M}$ there exists some $k \geq 0$ such that $m^k = m^{k+1}$; $\mathcal{M}$ is *group-free* if no subsemigroup of $\mathcal{M}$ is a nontrivial group. A finite monoid is group-free if and only if it is aperiodic. However, infinite monoids can be group-free and not aperiodic. The monoid $(\mathbb{N}, +, 0)$ is one such example. As we show below, infinite stabilizers of infinite words supply another natural example of this phenomenon.

Proposition 15. *Let $\mathbf{w} \in \Sigma^\omega$ be an aperiodic infinite word. Then $\mathcal{FS}tab(\mathbf{w})$ is an aperiodic monoid.*

Proof. Since $\mathbf{w}$ is aperiodic, $\mathcal{FS}tab(\mathbf{w})$ is a monoid. Let $\sigma = |\Sigma|$. Then $h^\sigma = h^{\sigma+1}$ for all $h \in \mathcal{FS}tab(\mathbf{w})$. □

Proposition 16. *Let $\mathbf{w} \in \Sigma^\omega$. Then $\mathcal{S}tab(\mathbf{w})$ is group-free, and $\mathcal{S}tab(\mathbf{w})$ is aperiodic if and only if $\mathcal{MS}tab = \emptyset$.*

Proof. Denote $\sigma = |\Sigma|$. Let $\mathcal{G}$ be a subsemigroup of $\mathcal{S}tab(\mathbf{w})$, and suppose that $\mathcal{G}$ is a group. Let $e \in \mathcal{G}$ be the unit element. Then $e^2 = e$, and so $e \in \mathcal{FS}tab(\mathbf{w})$, and $e(a) = \varepsilon$ for all $a \in M_e$. Let $h \in \mathcal{G}$, and let $g = h^{-1} \in \mathcal{G}$. Then for all $a \in M_e$ we have $h(a) = he(a) = h(\varepsilon) = \varepsilon$, and so $M_e \subseteq M_h$; and for all $a \in M_h$ we have $e(a) = (g^\sigma h^\sigma)(a) = g^\sigma(\varepsilon) = \varepsilon$, and so $M_h \subseteq M_e$. We get that for all $h \in \mathcal{G}$, $M_h = M_e = M$, and $h(a) = \varepsilon$ for all $a \in M$.

Let MR_e, MR_h be the sets of the monorecursive letters of e and h, respectively. Since $e \in \mathcal{FS}tab(\mathbf{w})$, $MR_e = \Sigma \setminus M$, and so $MR_h \subseteq MR_e$. Suppose there exists $b \in MR_e \setminus MR_h$. Then $e(b) = xby$ for some $x, y \in M^*$, while $h(b) = ucvdw$, where $u, v, w \in \Sigma^*$ and $c, d \in \Sigma \setminus M$. But then $xby = e(b) = gh(b) = g(ucvdw)$, a contradiction: since c, d are immortal with respect to g as well, $g(ucvdw)$ contains at least two letters of $\Sigma \setminus M$. Therefore, for all $h \in \mathcal{G}$, $MR_h = MR_e = MR$ and $h \in \mathcal{FS}tab(\mathbf{w})$.

Let $a \in \Sigma$. If $a \in M$, then $h(a) = e(a) = \varepsilon$. Otherwise, $h(a) = xay$, where $x, y \in M^*$, and $h(x) = h(y) = \varepsilon$. Since $M = M_g$ and $g(b) = \varepsilon$ for all $b \in M$, we get that $e(a) = gh(a) = g(xay) = g(a)$, and so $e(a) = hg(a) = he(a) = h(a)$. We conclude that $h(a) = e(a)$ for all $a \in \Sigma$, and so $h = e$ for all $h \in \mathcal{G}$. Thus $\mathcal{S}tab(\mathbf{w})$ is group-free.

For the second assertion, observe that if $\mathcal{MS}tab(\mathbf{w}) = \emptyset$ then $\mathcal{S}tab(\mathbf{w}) = \mathcal{FS}tab(\mathbf{w})$. In particular, $\mathcal{FS}tab(\mathbf{w})$ is a monoid (we note that if $\mathcal{MS}tab(\mathbf{w}) = \emptyset$ then necessarily $\mathbf{w}$ is an aperiodic infinite word, and so $\mathcal{FS}tab(\mathbf{w})$ is a monoid by Proposition 10 as well). By Proposition 15, this monoid ia aperiodic. On the other hand, if $\mathcal{MS}tab(\mathbf{w})$ contains a morphism h, then $h^n \neq h^m$ for all $n \neq m$ by Proposition 3, and so $\mathcal{S}tab(\mathbf{w})$ is not aperiodic. □

Below is a summary of the properties derived in this section.

1. Let $\mathbf{w} \in \Sigma^\omega$ be any infinite word, possibly ultimately periodic.
   (a) $\mathcal{S}tab(\mathbf{w})$ is a disjoint union of $\mathcal{FS}tab$ and $\mathcal{MS}tab$.
   (b) $\mathcal{FS}tab(\mathbf{w})$ is a finite nonempty set; $\mathcal{MS}tab$ is either empty or infinite.
   (c) $h \in \mathcal{MS}tab(\mathbf{w})$ if and only if $h \in \mathcal{S}tab(\mathbf{w})$, and there exists a letter $a \in \Sigma$ such that $h(a)$ contains at least two immortal letters.

(d) $h \in \mathcal{MS}tab(\mathbf{w})$ if and only if $h \in \mathcal{S}tab(\mathbf{w})$, and $|h(p)| > |p|$ for all sufficiently long prefixes p of $\mathbf{w}$.
(e) $\mathcal{MS}tab(\mathbf{w})$ is a subsemigroup of $\mathcal{S}tab(\mathbf{w})$.
(f) $\mathcal{S}tab(\mathbf{w})$ is a group-free monoid.

2. Let $\mathbf{w} \in \Sigma^\omega$ be an aperiodic infinite word.
   (a) $h \in \mathcal{MS}tab(\mathbf{w})$ if and only if $h \in \mathcal{S}tab(\mathbf{w})$, and the prefixes p of $\mathbf{w}$ satisfy $\lim_{|p|\to\infty}(|h(p)| - |p|) = \infty$.
   (b) $\mathcal{FS}tab(\mathbf{w})$ is an aperiodic submonoid of $\mathcal{S}tab(\mathbf{w})$.
   (c) $\mathcal{MS}tab(\mathbf{w})$ is an ideal of $\mathcal{S}tab(\mathbf{w})$.
   (d) $\mathcal{S}tab(\mathbf{w})$ is an aperiodic monoid if and only if $\mathcal{MS}tab(\mathbf{w}) = \emptyset$, if and only if $\mathcal{S}tab(\mathbf{w})$ is finite.
   (e) $\mathcal{S}tab(\mathbf{w})$ is finitely generated if and only if $\mathcal{MS}tab(\mathbf{w})$ is finitely generated.
   (f) $\mathcal{S}tab(\mathbf{w})$ can be infinitely generated.

4. Order of growth

Let Σ be a finite alphabet. The *order of growth* of a letter $a \in \Sigma$ under a morphism $h : \Sigma^* \to \Sigma^*$ is the function $\rho_{a,h} : \mathbb{N} \to \mathbb{N}$ defined by $\rho_{a,h}(n) = |h^n(a)|$, $n \in \mathbb{N}$. In [16, Theorem 4.14], Vitányi proved that there are only 4 possible growth types:

1. a is *exponentially growing* if $\limsup_{n\to\infty} |h^n(a)|/r^n > 0$ for some $r > 1$;
2. a is *polynomially growing* if there exist polynomials p and q of positive degree such that $p(n) \le |h^n(a)| \le q(n)$ for all $n \in \mathbb{N}$;
3. a is *limited* if there exists a constant C such that $0 < |h^n(a)| < C$ for all $n \in \mathbb{N}$;
4. a is *mortal* if $|h^n(a)| = 0$ for some $n \ge 0$.

If a belongs to one of the first two types we say that a is *growing under* h; otherwise, a is *bounded under* h. The set of growing letters is denoted by G_h; the set of bounded letters is denoted by B_h. Note that by definition, $h(a) \in B_h^*$ for all $a \in B_h$, and $h(b)$ contains a growing letter for all $b \in G_h$.

Let $\mathbf{w} \in \Sigma^\omega$, and let $h \in \mathcal{MS}tab(\mathbf{w})$. Then $\mathbf{w} = vh^\omega(a)$, where $v \in \Sigma^*$ is a finite fixed point of h and $a \in \Sigma$ satisfies $h(a) = yax$, with y mortal and x immortal. The *order of growth of* $\mathbf{w}$ *under* h is defined by $\rho_{\mathbf{w},h}(n) = |vh^n(a)| = |v| + \rho_{a,h}(n)$, where a is the first growing letter of $\mathbf{w}$ (note that there must be a growing letter, since $\mathbf{w}$ is infinite). We say that h is an exponentially (resp. polynomially) growing morphism if $\mathbf{w}$ grows exponentially (resp. polynomially) under h.

Another way to see that the only possible growth types for $\mathbf{w}$ are polynomial or exponential is through the incidence matrix. Let $\Sigma = \Sigma_k = \{0, 1, \ldots, k-1\}$, let $h : \Sigma_k^* \to \Sigma_k^*$, and let $u \in \Sigma_k^*$. The *Parikh vector* of u, denoted by $[u]$, is a vector of size k that counts how many times different letters occur in u: $[u] = (|u|_0, |u|_1, \ldots, |u|_{k-1})^T$. The *incidence matrix* associated with h, denoted by $A(h)$, is a $k \times k$ matrix, whose jth column is the Parikh vector of $h(j)$:

$$A(h) = (a_{i,j})_{0\le i,j<k} \ ; \ \ a_{i,j} = |h(j)|_i \ .$$

Let $A = A(h)$. It is a straightforward induction to show that $[h(u)] = A[u]$ for all $u \in \Sigma^*$, and that $A(h^n) = A^n$ for all $n > 0$. This implies that if $\mathbf{w} = vh^\omega(a)$, then $|vh^n(a)| = |v| + \mathbf{1}A^n[a]$, where $\mathbf{1} = (1, 1, \ldots, 1)$ (the all ones vector of size n). Thus, the growth order of $\mathbf{w}$ under h is the same as the growth order of $\|A^n\|$, where for a matrix $B = (b_{ij})$, $\|B\| = \max_{i,j} |b_{ij}|$. It is well known that $\|A^n\| \in \Theta(r^n n^{d-1})$, where r is the Perron-Frobenius eigenvalue of A, and d is the size of the largest Jordan block associated with r (see, e.g., [1, Chapter 8]; [9, 13]). Thus, $\mathbf{w}$ grows polynomially under h if and only if $r = 1$, and exponentially if and only if $r > 1$ (note that $r \geq 1$, otherwise $\mathbf{w}$ would be finite). In particular, if the growth is polynomial then the degree of polynomial is a natural number.

Our main goal in this section is to prove that for every aperiodic infinite word $\mathbf{w} \in \Sigma^\omega$, the elements of $\mathcal{MS}tab(\mathbf{w})$ can grow either exponentially or polynomially, but the two growth types cannot exist simultaneously. Moreover, in the polynomial case the degree of the polynomial is fixed for all $h \in \mathcal{MS}tab(\mathbf{w})$. We also show that we can compute the polynomial degree from the dependency graph of h. The proof is self-contained, and does not use the references mentioned above. Our results also enable us to extend Durand's generalization of Cobham's theorem [2, 3] to a wider family of morphisms.

For some of the proofs, it will be convenient to consider a power of h instead of h itself. Clearly, if a letter a is exponentially growing (resp. polynomially growing, limited, mortal) under a morphism h then it does so under the morphism h^t for all $t \geq 1$. The polynomial degree in the case of polynomial growth does not change either, since if $\left| h^n(a) \right| \in \theta(n^d)$, then $\left| h^{tn}(a) \right| \in \theta(t^d n^d) = \theta(n^d)$ for any constant t. Therefore, we can replace h by any convenient power of h. This leads to the following definition:

Definition3. A morphism $h : \Sigma^* \to \Sigma^*$ is *normalized* if it satisfies following conditions:

1. $\mathrm{alph}(h(a)) = \mathrm{alph}(h^2(a))$ for all $a \in \Sigma$;
2. the first (resp. last) growing letter in $h(a)$ is identical to the first (resp. last) growing letter in $h^2(a)$ for all $a \in G_h$;
3. $h(a) = h^2(a)$ for all $a \in B_h$.

The application is as follows.

Lemma 17. *Let $h : \Sigma^* \to \Sigma^*$ be a morphism. Then there exists a power $g = h^t$ with $t \geq 1$ such that g is normalized.*

Proof. Let 2^Σ be the power set of Σ. Then h induces a mapping $\bar{h} : 2^\Sigma \to 2^\Sigma$, defined by

$$\bar{h}(A) = \bigcup_{a \in A} \mathrm{alph}(h(a)), \quad A \in 2^\Sigma.$$

As the set of mappings $2^\Sigma \to 2^\Sigma$ is a finite monoid, there exists an integer $t \geq 1$ such that $\bar{h}^t = (\bar{h}^t)^2$. This implies that the morphism $h_1 = h^t$ satisfies (1); and we may henceforth assume $h_1 = h$.

Now, for a morphism h and a growing letter a let $\hat{a}$ be the first growing letter in $h(a)$, and define a mapping $\bar{h} : G_h \to G_h$ by $\bar{h}(a) = \hat{a}$. Clearly, $\overline{h^t} = (\bar{h})^t$. By the same argument (some power of $\bar{h}$ becomes idempotent) we get that there exists some power h_2 of $h_1 = h$ such that the first growing letter in $h_2(a)$ is identical to the first growing letter in $h_2^2(a)$ for all $a \in G_h$. Similarly, there exists some power h_3 of h_2 such that the last growing letter in $h_3(a)$ is identical to the last growing letter in $h_3^2(a)$ for all $a \in G_h$.

Finally, let $Q = \{h_3^k(a) : \ a \in B_h,\ k \geq 0\}$, and define $\bar{h}_3 : Q \to Q$ by $\bar{h}_3(x) = h_3(x)$ for all $x \in Q$. Since Q is a finite set, we get by the same reasoning that there exists some power g of h_3 that satisfies (1), (2) and (3). $\square$

By Lemma 17, we may assume whenever it is convenient that h is normalized. In particular, we may assume $h(a) = \varepsilon$ for all $a \in M_h$, and $h(x) = h^2(x)$ for all $x \in B_h^*$.

4.1. The edge condition

The following definition is due to Ehrenfeucht and Rozenberg [5]:

Definition4. Let $h : \Sigma^* \to \Sigma^*$. A letter $a \in \Sigma$ satisfies the *edge condition* under h if there exists an integer t such that $h^t(a) = xay$ (or $h^t(a) = yax$), where $x \in \Sigma^*$ and $y \in B_h^* \setminus M_h^*$. Note that if h is normalized, then a satisfies the edge condition under h if and only if it satisfies it with $t = 1$.

The edge condition is a key concept for analyzing the order of growth. Let $\mathbf{w} = vh^\omega(a) \in \Sigma^\omega$ be an aperiodic infinite word. In what follows, we use two classical results: one, due to Pansiot [12], states that there exists a letter satisfying the edge condition under h if and only if the *factor complexity* of $\mathbf{w}$ is quadratic; the other, due to Ehrenfeucht and Rozenberg [5], states that there exists a letter satisfying the edge condition under h if and only if $\mathbf{w}$ contains *unbounded powers of bounded letters*. By showing that polynomial growth implies the existence of a letter satisfying the edge condition, we are able to combine those two results.

4.2. Factor complexity

Pansiot's result [12] relates the factor complexity of $\mathbf{w}$ to the existence of a letter satisfying the edge condition. In fact, we only use Corollary 20 and for this we do not need the full statement of Pansiot's results; Propositions 18 and 19 are enough. For the sake of completeness we give full proofs. Readers familiar with Pansiot's results are invited to proceed directly with Corollary 20.

Definition5. Let $\mathbf{w} \in \Sigma^\omega$ be an infinite word. The *factor complexity* (also called *subword complexity* elsewhere) of $\mathbf{w}$ is the function $p_{\mathbf{w}} : \mathbb{N} \to \mathbb{N}$ defined by $p_{\mathbf{w}}(n) = |\mathrm{Fact}(\mathbf{w}) \cap \Sigma^n|$. That is, $p_{\mathbf{w}}(n)$ counts the number of distinct factors of length n of $\mathbf{w}$.

Proposition 18 (Pansiot [12]). *Let $\mathbf{w} = vh^\omega(a) \in \Sigma^\omega$ be an aperiodic infinite word. If some letter $b \in \Sigma$ satisfies the edge condition, then $p_{\mathbf{w}}(n) \in \Omega(n^2)$.*

PROOF. We assume that h is normalized. By assumption, $h(a) = yax$, where y is mortal and x immortal, and $\mathbf{w} = v'axh(x)h^2(x)\cdots$, where $v' = vh^{|\Sigma|}(y)\cdots y$. Suppose that $b \in \Sigma$ satisfies the edge condition. Then $b \neq a$, else we would get that $x \in B_h^*$, and $\mathbf{w} = v'ax(h(x))^\omega$, a contradiction. Since $b \in \mathrm{alph}(h(a))$ and $b \in \mathrm{alph}(h(b))$, we get that b occurs infinitely often in $\mathbf{w}$. By symmetry arguments, we may assume that $h(b) = rbs$, where $r \in \Sigma^*$ and $s \in B_h^* \setminus M_h^*$.

Let $u'bz'c'$ be the prefix of $\mathbf{w}$ such that b does not occur in u', $z' \in B_h^*$, and c' is growing. Then $h(u'bz'c') = h(u')rbsh(z')h(c')$. Let $h(c') = y'cy$, where $y' \in B_h^*$ (that is, c is the first growing letter of $h(c')$). Let $u = h(u')r$, and let $z = sh(z')y'$. Since h is normalized, c is the first growing letter of $h(c)$. We get that $\mathbf{w}$ has a prefix of the form $ubzc$, where $|u| > |vh(a)|$, $z \in B_h^*$, c is growing, and $h(c) \in yc\Sigma^*$ for some $y \in B_h^*$. In particular, $b, c \notin \mathrm{alph}(z)$.

For all $k \in \mathbb{N}$, consider the prefix

$$h^k(ubzc) = h^k(u)\cdot h^{k-1}(r)\cdots h(r)r\cdot b\cdot s\cdot (h(s))^{k-1}\cdot h(z)\cdot (h(y))^{k-1}\cdot y\cdot c := u_k b z_k c,$$

where $u_k = h^k(u)h^{k-1}(r)\cdots h(r)r$ and $z_k = s(h(s))^{k-1}h(z)(h(y))^{k-1}y \in B_h^*$. Then $|z_k| \in \Theta(k)$, and so the number of z_k with $n/2 < |z_k| < 3n/4$ is of order n. Also, since $|u| > 1$ and u contains at least 2 growing letters (namely, a and b), we get by induction that $|u_k| > k$. This implies that for each k, $\mathbf{w}$ contains a factor of the form bz_kc, with at least k letters strictly to the left of this factor. Hence, for $n/2 < |z_k|$, we can assume that $k > n/C$ for some constant $C \geq 4$.

The fraction $1/C$ is used to guarantee for each $0 \leq \ell \leq n/C$ and each k with $n/2 < |z_k| < 3n/4$ the existence of a factor $s_{k,\ell}bz_kcp_{k,\ell}$ in $\mathbf{w}$, where $|s_{k,\ell}| = \ell$ and $|s_{k,\ell}bz_kcp_{k,\ell}| = n$.

Note that if $n/2 < |z_k| < |z_m|$, then bz_kc and bz_mc cannot occur in the same factor of $\mathbf{w}$ of length n; The reason is that b and c are growing letters which do not occur inside z_k. Now for $0 \leq \ell < \ell' \leq n/C$ the words $s_{k,\ell}bz_kcp_{k,\ell}$ and $s_{k,\ell'}bz_kcp_{k,\ell'}$ are different as well.

We have thus found $\Omega(n^2)$ different factors of length n in $\mathbf{w}$. □

Proposition 19 (Pansiot [12]). *Let $\mathbf{w} = vh^\omega(a) \in \Sigma^\omega$ be an aperiodic infinite word. If no letter satisfies the edge condition, then $p_{\mathbf{w}}(n) \in O(n \log n)$.*

PROOF. We assume that h is normalized. Moreover, since no letter a satisfies the edge condition, each letter is either bounded or exponentially growing. Replacing h by some power, we may assume that for all $b \in \Sigma$ and for all $\ell \in \mathbb{N}$, either $h(b) = h^2(b)$ or $\left| h^\ell(b) \right| \geq 2^\ell$. Let w be a factor of length n in $\mathbf{w}$, where $n > \left| vh^4(a) \right|$. We count the number of factors of this length.

Let ℓ be the minimal integer such that $u := h^{-(\ell+2)}(w)$ is a factor of $vh^2(a)$. Then $|u| \geq 2$: otherwise, if $|u| = 1$, we get that $h(u) \in \mathrm{Fact}(vh^2(a))$ (recall that h is normalized), and so $h^{-(\ell+1)}(w) \in \mathrm{Fact}(vh^2(a))$, a contradiction to the minimality of ℓ. Therefore, $u = a_1 \cdots a_m$, with $a_i \in \Sigma$ and $2 \leq m \leq |vh^2(a)|$. There is only a constant number of such u's, and so we may fix u. Note that w factorizes as $sh^{\ell+2}(a_2)\cdots h^{\ell+2}(a_{m-1})p$, where s is a suffix of $h^{\ell+2}(a_1)$ and p is a prefix of $h^{\ell+2}(a_m)$. For a fixed n, w is completely described by u, $|s|$, and ℓ, where $1 \le |s| \le n$. Therefore it is enough to show that $\ell \le \log(n)$, where (in this proof) $\log(n)$ is simply a short hand for the positive integer $\lceil \log_2 n \rceil$.

If a_i is growing for some $2 \le i \le m-1$, then $|h^{\ell+2}(a_i)| \ge 2^\ell$, and so $\ell \le \log n$. Assume therefore that $a_i \in B_h$ for $2 \le i \le m-1$. Let $u = a_1 z a_m$, where $z \in B_h^*$. Then $h(z) = h^2(z)$. Since $|w| > \left| vh^4(a) \right|$ and $h^{\ell+2}(u)$ contains w, at least one of a_1, a_m is growing. By symmetry we assume that a_1 is growing. Let $h(a_1) = x'bx''$, where b is the last growing letter of $h(a_1)$. Since h is normalized, $h(b) = xbz'$, where $x \in \Sigma^*$ and $z' \in B_h^*$. But as b does not satisfy the edge condition, necessarily z' is mortal. Hence $h^\ell(xbz') = h^\ell(x) \cdots h(x)xbz'$.

For a_m we write $h(a_m) = y''cy'$, where $cy' = \varepsilon$ if $h(a_m) \in B_h^*$, and c is the first growing letter of $h(a_m)$ otherwise. Since h is normalized, we can assume that $h(c) = z''cy$, where z'' is mortal (if a_m is bounded, then $h(c) = \varepsilon$ and the equality still holds).

Recall that

$$h^2(u) = h(x'bx''h(z)y''cy') = h(x')xbz'h(x''zy'')z''cyh(y').$$

Assume that $\ell > \log n$. We show that this is impossible: Since $x''h(z)y'' \in B_h^*$ we have

$$h(bz'x''h(z)y''z''c) = xbz'h(x''zy'')z''cy,$$

and by induction,

$$h^{\ell+1}(bz'x''h(z)y''z''c) = h^\ell(x) \cdots h(x)xbz'h(x''zy'')z''cyh(y) \cdots h^\ell(y).$$

We get that s is a suffix of $h^{\log(n)}(x) \cdots h(x)xbz'h(x'')$, because the length of $h^{\log(n)}(xbz') = h^{\log(n)}(x) \cdots h(x)xbz'$ exceeds n and $h^{\log(n)}(x) \cdots h(x)xbz'h(x'')$ is a suffix of $h^\ell(x) \cdots h(x)xbz'h(x'')$.

Consider now $h(y'')z''cyh(y) \cdots h^\ell(y)$. If a_m is bounded, then p is a prefix of $h(y'')$ because the other parts are empty. In particular, p is also a prefix of $h(y'')z''cyh(y) \cdots h^{\log(n)}(y)$ in this case.

If on the other hand a_m is growing, then a dual argument as above shows that p is a prefix of $h(y'')z''cyh(y) \cdots h^{\log(n)}(y)$, because c grows in this case. Thus, whether or not a_m is growing, p is a prefix of $h(y'')z''cyh(y) \cdots h^{\log(n)}(y)$. But this contradicts the minimality of ℓ, because w appears as a factor in $h^{\log(n)}(u) = h^{\log(n}(a_1 \cdots a_m)$. Therefore $\ell \le \log n \le 1 + \log_2 n$. □

Corollary 20. *Let $\mathbf{w} \in \Sigma^\omega$ be an aperiodic infinite word, and suppose that there exists a morphism $h \in \mathcal{MS}tab(\mathbf{w})$ and a letter $a \in \Sigma$ such that a satisfies the edge condition under h. Then for all $g \in \mathcal{MS}tab(\mathbf{w})$ there exists a letter $b \in \Sigma$ such that b satisfies the edge condition under g.*

Proof. By the above propositions, for any morphism $h \in \mathcal{MS}tab(\mathbf{w})$ there exists a letter satisfying the edge condition under h if and only if $p_{\mathbf{w}}(n) \in \Omega(n^2)$. As the factor complexity is independent of the generating morphism, the result follows. □

4.3. Unbounded powers over B_h^*

Let $\mathbf{w} \in \Sigma^\omega$. To differentiate between exponentially growing morphisms and polynomially growing morphisms in $\mathcal{MS}tab(\mathbf{w})$, we analyze the structure of unbounded powers u^i occurring in $\mathbf{w}$, where $u \in B_h^*$. First, we consider aperiodic infinite words in general.

The following lemma is due to Ehrenfeucht and Rozenberg [5]. We bring the proof (slightly adjusted) as it will be used next. In what follows, the *inverse image* $h^{-1}(u)$ of a word u occurring in $\mathbf{w} = vh^\omega(a)$ is the shortest occurrence u' in $\mathbf{w}$ such that $h(u')$ contains u.

Lemma 21 ([5]). *Let $\mathbf{w} = vh^\omega(a) \in \Sigma^\omega$. The following are equivalent:*

1. *there exists a letter $b \in \Sigma$ that satisfies the edge condition;*
2. *there exists a nonempty word $z \in B_h^+$ such that $z^n \in Fact(\mathbf{w})$ for all $n \in \mathbb{N}$.*

Proof. We assume that h is normalized. Suppose there exists a letter $b \in \Sigma$ that satisfies the edge condition. Assume that $h(b) = xby$, where y is a bounded immortal word. Since h is normalized, $h(y) = h^2(y)$. Denote $x_n = h^{n-1}(x) \cdots h^n(x)x$ and $z = h(y)$. Then for all $n \in \mathbb{N}$, the word $h^n(b) = x_n byz^{n-1}$ is a factor of $\mathbf{w}$.

Now suppose that there exists a word $z \in B_h^+$ such that $z^n \in \mathrm{Fact}(\mathbf{w})$ for all $n \in \mathbb{N}$. Then $\mathbf{w}$ contains arbitrarily long factors of bounded letters. First, we assume that $\mathbf{w}$ contains infinitely many growing letters. Choose $b_0y_0c_0 \in \mathrm{Fact}(\mathbf{w})$, where $b_0, c_0 \in G_h$, $y_0 \in B_h^*$, and $|y_0| > 3\|h\|^2 + 2\|h\|$ (note: though for taking an inverse image of y_0 it is enough to require $|y_0| > \|h\|$, the condition $|y_0| > 3\|h\|^2 + 2\|h\|$ is needed to derive a contradiction at the end).

Consider $h^{-1}(b_0y_0c_0)$. Since $|y_0| > \|h\|$, and since both $h^{-1}(b), h^{-1}(c)$ are growing letters, we get that $h^{-1}(b_0y_0c_0) = b_1y_1c_1$, where $b_1, c_1 \in G_h$ and $y_1 \in B_h^*$. Also, b_0 is the last growing letter in $h(b_1)$ and c_0 is the first growing letter in $h(c_1)$. This implies that $|y_0| < \|h\||y_1| + 2\|h\|$. Together with $|y_0| > 3\|h\|^2 + 2\|h\|$, we get that $|y_1| > 3\|h\|$, and we can apply inverse image again. We get that $h^{-1}(b_1y_1c_1) = b_2y_2c_2$, where $b_2, c_2 \in G_h$, $y_2 \in B_h^*$, and $|y_2| > 0$.

The successive application of h^{-1} can be continued as long as $|y_i| \geq \|h\|$. By induction, we get a sequence $\{b_iy_ic_i\}_{0 \leq i \leq k}$, where $k \geq 2$, $b_i, c_i \in G_h$ and $y_i \in B_h^*$ for $0 \leq i \leq k$, and for $0 \leq i \leq k-1$,

- $b_{i+1}y_{i+1}c_{i+1} = h^{-1}(b_iy_ic_i)$;
- b_i is the last growing letter of $h(b_{i+1})$;
- c_i is the first growing letter of $h(c_{i+1})$;
- $\|h\| \leq |y_i| < \|h\||y_{i+1}| + 2\|h\|$;
- $|y_k| < \|h\|$.

Since h is normalized, we get that $b_{k-1} = b_i = b$ and $c_{k-1} = c_i = c$ for all $0 \le i < k$. Let $h(b_k) = u'bx'$ and $h(c_k) = z'cv'$, where $x', z' \in B_h^*$ and $u', v' \in \Sigma^*$. Then

$$h(b_k y_k c_k) = u'bx'h(y_k)z'cv', \quad \text{and} \quad y_{k-1} = x'h(y_k)z'.$$

Suppose neither b nor c satisfy the edge condition. This implies that $h(b) = ubx$ and $h(c) = zcv$, where $u, v \in \Sigma^*$ and $x, z \in M_h^*$. Since h is normalized, $h(x) = h(z) = \varepsilon$ and $h(w) = h^2(w)$ for all $w \in B_h^*$ (in particular, for $w = x', z', y_k$). We get:

$$\begin{array}{rclcl} h(b_{k-1}y_{k-1}c_{k-1}) & = & h(bx'h(y_k)z'c) & = & ubxh(x'y_kz')zcv; \\ y_{k-2} & = & xh(x'y_kz')z; & & \\ h(b_{k-2}y_{k-2}c_{k-2}) & = & h(bxh(x'y_kz')zc) & = & ubxh(x'y_kz')zcv; \\ y_{k-3} & = & xh(x'y_kz')z; & & \end{array}$$

and similarly, $y_i = xh(x'y_kz')z$ for all $0 \le i \le k-2$. Since $|x|, |x'|, |z|, |z'|, |y_k| < \|h\|$, we get that $|y_0| < 3\|h\|^2 + 2\|h\|$, a contradiction to how we chose y_0. Therefore, at least one of b, c satisfies the edge condition.

If $\mathbf{w}$ contains only finitely many growing letters $a_1, \ldots, a_k$ then necessarily $h^{-1}(a_i) = a_i$ for all $1 \le i \le k$. This implies that for all $b \in G_h$, the only growing letter in $h(b)$ is b itself. Since b is growing, it has to satisfy the edge condition. □

Definition6. A word $x \in \Sigma^+$ is *primitive* if there exist no word $y \in \Sigma^+$ and no integer $k \ge 2$ such that $x = y^n$. Let $\mathbf{w} \in \Sigma^\omega$. The *repetitive set* of $\mathbf{w}$ is the set

$$Rep(\mathbf{w}) = \{x \in \Sigma^+ : x \text{ is primitive and } x^k \in \mathrm{Fact}(\mathbf{w})\ \forall k \in \mathbb{N}\}.$$

Definition7. Let $\mathbf{w} = vh^\omega(a) \in \Sigma^\omega$. The *index of order n* of a word $x \in \Sigma^+$ under the morphism h is defined by

$$ind_{\mathbf{w},h}(x, n) = \max\{k \in \mathbb{N} : x^k \in \mathrm{Fact}(vh^n(a))\}.$$

Lemma 22. *Let $\mathbf{w} = vh^\omega(a) \in \Sigma^\omega$ be an aperiodic infinite word. Then we have $ind_{\mathbf{w},h}(u, n) \in \Theta(n)$ for all $u \in Rep(\mathbf{w}) \cap B_h^*$. Moreover, $Rep(\mathbf{w}) \cap B_h^*$ is given by the cyclic shifts of at most $2|G_h|$ words.*

Proof. Since $\mathbf{w}$ is aperiodic, it must contain infinitely many growing letters. Suppose there exists some $u \in Rep(\mathbf{w}) \cap B_h^*$, and consider a maximal power u^i (here u^i is maximal if it occurs at position ℓ in $\mathbf{w}$, but does not occur at position $\ell - |u|$ or $\ell + |u|$). Let $K = 3\|h\|^2 + 2\|h\|$. By the proof of Lemma 21, if $|u^i| > K$, then u^i is contained in a word of the form $h^k(bwc)$, where $w = x'h(y')z'$, and

- $b, c \in G_h$;
- $x', y', z' \in B_h^*$;
- $|x'|, |y'|, |z'| < \|h\|$;

- $h(b) \in \Sigma^* bx$ and $h(c) \in zc\Sigma^*$, where $x, z \in B_h^*$ and at least one of x, z is immortal.

Let $y = h(x'h(y')z') = h(x'y'z')$. Then u^i is a factor of $bx(h(x))^k y(h(z))^k zc$, where $x, z, y \in B_h^*$, $|x|, |z| < \|h\|$, and $|y| < 3\|h\|^2$. By choosing $i > 2K$, we get that there is a power u^K that is contained as a factor in $(h(x))^k$ or $(h(z))^k$ (or both). By symmetry, we assume that u^K occurs in $h(x)$ (and x is immortal). Since u is primitive, this implies that some cyclic shift of u is a primitive root of $h(x)$; this is a consequence of well-known theorems of Lyndon and Schützenberger [11]. Since the number of words $h(x)$ we have to consider is bounded by $2|G_h|$, this yields the finiteness of $Rep(\mathbf{w}) \cap B_h^*$. Also, since i was maximal, this implies that $i \geq k - 1$. Let n_0 be the minimal integer such that $h^{n_0}(a)$ contains all factors of the form bwc, with $b, c \in G_h$, $w \in B_h^*$, and $|w| < \|h\|^2 + 2\|h\|$. Then for sufficiently large n, $ind_{\mathbf{w},h}(u, n) \geq n - n_0 - 2$. This shows the lower bound.

For the upper bound, observe that if u^i is a factor of $bx(h(x))^k y(h(z))^k zc$, then $|u^i| \leq |x| + |y| + |z| + k(|h(x)| + |h(z)|) < K + 2\|h\|^2 k$. Therefore, the same upper bound applies to i. Suppose that u^i occurs in $h^n(a)$. Then $k \leq n$, and the upper bound follows. $\square$

Next, we consider aperiodic infinite words generated by polynomially growing morphisms.

Lemma 23. *Let $\mathbf{w} \in \Sigma^\omega$ be an aperiodic infinite word, and let $h \in \mathcal{MS}tab(\mathbf{w})$ be a normalized, polynomially growing morphism. Then for all $a \in G_h$, $h(a)$ contains a letter satisfying the edge condition.*

Proof. Since h is normalized, $\mathrm{alph}(h(a)) = \mathrm{alph}(h^2(a))$ for all $a \in \Sigma$. Let $a \neq b$, and suppose $b \in \mathrm{alph}(h(a))$. Then $a \notin \mathrm{alph}(h(b))$: otherwise, we would get that $h^2(a) = xayaz$ for some $x, y, z \in \Sigma^*$, which implies that a grows exponentially, a contradiction (see also Salomaa [14]). Therefore, the relation "$b \preceq a \Leftrightarrow b \in \mathrm{alph}(h(a))$" is a partial order on Σ. A least growing letter b under this order satisfies $h(b) = xby$, where $x, y \in B_h^*$ (or there would be a growing letter smaller than b), and xy is immortal (or b would not be growing). That is, b satisfies the edge condition. Therefore, for each growing letter a there exists some descending chain that begins from a and contains a letter b that satisfies the edge condition. Since h is normalized, $b \in \mathrm{alph}(h(a))$. $\square$

Proposition 24. *Let $\mathbf{w} = vh^\omega(a) \in \Sigma^\omega$ be an aperiodic infinite word such that h grows polynomially. Then we have $\emptyset \neq Rep(\mathbf{w}) \subseteq B_h^+$.*

Proof. We may assume that h is normalized. Since a is growing, we get by Lemma 23 that $h(a)$ contains a letter satisfying the edge condition. By Lemma 21, we get therefore that there exists a word $u \in B_h^+$ such that u^k is a factor of $\mathbf{w}$ for all $k \geq 1$. Thus, $Rep(\mathbf{w}) \neq \emptyset$.

Next, we show that no $u \in Rep(\mathbf{w})$ contains a growing letter. Suppose that there exists a factor $w \in \mathrm{Fact}(\mathbf{w})$ that contains $2\|h\| - 1$ or more growing letters, but no letter that satisfies the edge condition. Then $h^{-1}(w)$ must contain at least one growing letter b such that $h(b)$ is contained in w. But by Lemma 23, $h(b)$ contains a letter satisfying the edge condition, a contradiction. We conclude that any factor of $\mathbf{w}$ contains at most $2\|h\| - 2$ consecutive growing letters that do not satisfy the edge condition (here two growing letters w_i and w_j are consecutive in $\mathbf{w} = w_0 w_1 w_2 \cdots$ if $w_{i+1} \cdots w_{j-1} \in B_h^*$).

Suppose there exists a word $u \in \mathit{Rep}(\mathbf{w})$ that contains a growing letter c. Then $h^{-1}(u)$ must contain a growing letter as well. Also, there exists some $t \geq 1$ such that if a is a letter satisfying the edge condition, with $h(a) = xay$ (resp. $h(a) = yax$) and $y \in B_h^*$, then $h^t(a) = x'ay'$, where $y' \in B_h^*$ and $|y'| > |u|$. By the assumption, $u^n \in \mathrm{Fact}(\mathbf{w})$ for all $n \in \mathbb{N}$. Choose some integer n large enough such that $v_n := h^{-t}(u^n)$ contains $4\|h\|$ or more growing letters. Then v_n must contain a letter b, such that b satisfies the edge condition and $h^t(b)$ is contained in u^n. But that implies that u^n contains a factor longer than $|u|$ that contains no growing letters, a contradiction. Therefore, $\mathit{Rep}(\mathbf{w}) \subseteq B_h^*$. □

Remark. Let $\mathbf{w}$ be as in Proposition 24. Then Lemma 22 implies that $\mathit{Rep}(\mathbf{w})$ is a finite set, given by the cyclic shifts of at most $2|G_h| \leq 2|\Sigma| - 2$ primitive words. The work of Kobayashi and Otto [8, Section 5] implies that if $\mathbf{w}$ can be generated by an injective morphism, then $\mathit{Rep}(\mathbf{w})$ is also finite. We believe that $\mathit{Rep}(\mathbf{w})$ is finite whenever $\mathcal{MS}\mathit{tab}(\mathbf{w})$ is nonempty.

4.4. Polynomial and exponential growth cannot co-exist

Theorem 25. *Let $\mathbf{w} \in \Sigma^\omega$ be an aperiodic infinite word, where $\mathcal{MS}\mathit{tab}(\mathbf{w})$ is not empty. Then $\mathcal{MS}\mathit{tab}(\mathbf{w})$ contains either polynomially growing morphisms or exponentially growing morphisms, but not both. Moreover, in the polynomial case, the degree of the polynomial is fixed.*

Proof. Let $\mathbf{w} = vh^\omega(a)$ for some polynomially growing morphism h. Fix some d_1 such that $|vh^n(a)| \in \mathcal{O}(n^{d_1})$. Suppose that $\mathcal{MS}\mathit{tab}(\mathbf{w})$ contains another morphism g, with $\mathbf{w} = ug^\omega(b)$, such that $|ug^n(b)| \in \Omega(n^{d_2})$ and $d_1 < d_2$. Note that this assumption is verified if either g is exponentially growing or if the degree of the polynomially growing morphisms were not fixed.

By Lemma 23, there exists a letter satisfying the edge condition under h. By Corollary 20, there exists therefore a letter satisfying the edge condition under g. By Lemma 21, there exists some nonempty word $x \in \mathit{Rep}(\mathbf{w}) \cap B_g^*$. Since by Proposition 24 we have $\mathit{Rep}(\mathbf{w}) \subseteq B_h^+$, it follows $x \in B_h^+ \cap B_g^+$. Therefore, by Lemma 22, $\mathit{ind}_{\mathbf{w},h}(x, n) \in \Theta(n)$ and $\mathit{ind}_{\mathbf{w},g}(x, n) \in \Theta(n)$.

For $n \in \mathbb{N}$ we find $m \in \mathcal{O}(n^{d_1/d_2})$ such that $|ug^m(b)| \geq |vh^n(a)|$. The maximal exponent of x in $ug^m(b)$ is of order m. But $|ug^m(b)| \geq |vh^n(a)|$, and the maximal exponent of x in $vh^n(a)$ is of order n, a contradiction. Thus, a polynomially growing morphism and an exponentially growing morphism cannot co-exist in the same stabilizer and moreover, $\mathcal{MS}\mathit{tab}(\mathbf{w})$ cannot contain polynomially growing morphisms of different degrees. □

4.5. The dependency graph and polynomial degree

The *dependency graph* of a morphism $h : \Sigma^* \to \Sigma^*$ is a directed graph $D(h)$, whose vertices are the letters of Σ, and that contains a directed edge $a \to b$ if and only if $b \in \text{alph}(h(a))$. If a is *recursive* (that is, $h(a) \in \Sigma^* a \Sigma^*$), then $D(h)$ contains a self-loop $a \to a$. We assume that h is normalized, and so $D(h)$ is transitive. In this section we show that we can deduce from $D(h)$ what is the order of growth of a letter a under h, and if the letter grows polynomially, we can use $D(h)$ to compute the polynomial degree.

First, we note that a sufficient condition for a letter a to grow exponentially is that it is contained in a non-trivial strongly connected component. Indeed, if there exists letters $a \neq b$ such that $a \in \text{alph}(h(b))$ and $b \in \text{alph}(h(a))$, then $h^2(a)$ contains at least two occurrences of a. This condition is not necessary however, as is apparent from the case $h(a) = aa$.

To overcome the difficulty, we label by E all letters $a \in \Sigma$ such that a appears at least twice in $h(a)$. Now, a letter a is exponentially growing if and only if there is an edge from a either to a letter labeled with E or to a letter b which is contained in a non-trivial strongly connected component. If we remove these letters we obtain a partial order (since all connected components are now trivial) and there is no letter labeled by E. We claim that all remaining letters have either a polynomial or a bounded growth, and that the graph encodes the polynomial degree as well as some other interesting facts.

Notation: For each letter $a \in \Sigma$, we find a path $a = a_0 \to a_1 \to \cdots \to a_\ell$ for which the number of recursive letters is maximal. We denote by $d(a)$ the number of recursive letters on such a path. Note that if a is immortal then $d(a) > 0$.

Proposition 26. *Let $h : \Sigma^* \to \Sigma^*$, and suppose that a occurs at most once in $h(a)$ for all $a \in \Sigma$, and that $D(h)$ induces a partial order on Σ. Then*

1. *a has a polynomial growth of degree $d(a) - 1$, where -1 is the degree of the zero polynomial;*
2. *Suppose $h(a) \in M_h^* a \Sigma^*$ (that is, iterating h on a gives a converging sequence). Then $d(a) \geq 1$, and*
   - *if $d(a) = 1$ then $h^\omega(a)$ is a finite word;*
   - *if $d(a) = 2$ then $h^\omega(a)$ is an ultimately periodic infinite word;*
   - *if $d(a) \geq 3$ then $h^\omega(a)$ an aperiodic infinite word.*

Proof. **Proof of 1.** First, consider the mortal and bounded letters. A letter a has growth order zero if and only if a is mortal, if and only if there are no outgoing edges from a, if and only if $d(a) = 0$. Thus, a has degree -1 if and only if $d(a) = 0$.

An letter a has polynomial growth with degree 0 if and only if a has a non-zero bounded growth, if and only if $h(a) = h^2(a) \neq \varepsilon$, if and only if $d(a) = 1$.

Now let $d(a) \geq 2$ and let $b = a_1$, where $a = a_0 \to a_1 \to \cdots \to a_\ell$ is a path containing $d(a)$ recursive letters. Then the degree of $a \geq$ the degree of b. If a is not recursive then $d(a) = d(b)$. By induction, b has degree $d(b) - 1$, and so the degree of a is at least $d(a) - 1$.

If a is recursive, then $\left| h^k(a) \right| \geq \left| h^{k-1}(a) \right| + \left| h^{k-1}(b) \right|$, that is, $\left| h^k(a) \right| - \left| h^{k-1}(a) \right| \geq \left| h^{k-1}(b) \right|$. Thus,

$$\left| h^k(a) \right| - 1 = \left| h^k(a) \right| - \left| h^{k-1}(a) \right| + \left| h^{k-1}(a) \right| - \cdots - \left| h(a) \right| + \left| h(a) \right| - |a| \geq \sum_{i=1}^{k-1} \left| h^i(b) \right|.$$

By induction, the degree of b is at least $d(b) - 1$, thus the degree of a is at least $d(b) = d(a) - 1$.

This yields the lower bound. For the upper bound we can argue in a very similar way. We let $h(a) = b_1 \cdots b_m$ and we assume that the growth of some $b = b_j$ is the fastest. Then $d(b) \leq d(a)$. If a is not recursive then $\left| h^k(a) \right| \leq m\left| h^{k-1}(b) \right|$. By induction, the degree of b is at most $d(b) - 1$, and so the degree of a is at most $d(a) - 1$. If a is recursive, then

$$\left| h^k(a) \right| - \left| h^{k-1}(a) \right| \leq m\left| h^{k-1}(b) \right|.$$

We obtain the result by induction on d.

Proof of 2. Let $h(a) = yax$, where $y \in M_h^*$ and $x \in \Sigma^*$. If $d(a) = 1$ then $x \in M_h^*$ as well, and $h^\omega(a)$ is finite. If $d(a) = 2$ then $x \in B_h^* \setminus M_h^*$, and $h^\omega(a) = yax(h(x))^\omega$.

Suppose $d(a) \geq 3$. Then x contains a recursive letter b with $d(b) = 2$ (take b to be the appropriate letter along the longest path beginning with a). Thus $h(b) = zbz'$, where $zz' \in B_h^* \setminus M_h^*$. We get that b satisfies the edge condition, and so $h^\omega(a)$ contains unbounded powers of the form u^i, where $u \in B_h^* \setminus M_h^*$. On the other hand, $h^\omega(a)$ contains infinitely many b's, and b is growing. Hence b does not occur in u^i, and $h^\omega(a)$ is aperiodic. $\square$

4.6. Multiplicative dependency of dominant eigenvalue

In [2], Durand made the following conjecture: let $\mathbf{w} \in \Sigma^\omega$ be an aperiodic infinite word, such that $\mathbf{w} = h^\omega(a) = g^\omega(a)$, where $h, g \in \mathcal{MS}tab(\mathbf{w})$ and a is the first letter of $\mathbf{w}$. Let $r(h)$ and $r(g)$ be the dominating eigenvalues of h and g, respectively. Then $r(h)$ and $r(g)$ are *multiplicatively dependent*. That is, there exist some integers n, m such that $r(h)^n = r(g)^m$.

Durand stated this conjecture as a generalization of Cobham's theorem in [2, 3]. In those papers he considered only *substitutions*, that is, morphisms $h : \Sigma^* \to \Sigma^*$ that satisfy $\lim_{n\to\infty} |h^n(a)| = \infty$ for all $a \in \Sigma$. In particular, such morphisms grow exponentially. Durand proved his conjecture for primitive substitutions in [2], and for a wide family of non-primitive substitutions in [3]. Theorem 25 extends the conjecture to polynomially growing morphisms. Moreover, it shows a stronger property: if the stabilizer elements grow polynomially, then the largest Jordan block associated with the dominating eigenvalue $r = 1$ is of a fixed size. More formally:

Theorem 27. *Let* $\mathbf{w} \in \Sigma^\omega$ *be an aperiodic infinite word, where* $\mathcal{MS}tab(\mathbf{w})$ *is not empty. For a morphism* $h \in \mathcal{MS}tab(\mathbf{w})$*, let* $r(h)$ *be the Perron-Frobenius eigenvalue of the incidence matrix of* h*, and let* $d(h)$ *be the size of the largest Jordan block associated with* $r(h)$*. Suppose there exists a morphism* $h \in \mathcal{MS}tab(\mathbf{w})$ *such that* $r(h) = 1$*. Then for all* $g \in \mathcal{MS}tab(\mathbf{w})$*,* $r(g) = r(h) = 1$ *and* $d(g) = d(h) \geq 3$*.*

Proof. This is an immediate result of Theorem 25, since $r(h) = 1$ if and only if h grows polynomially, and the degree of the polynomial is $d(h) - 1$. The inequality $d(h) \geq 3$ follows from Proposition 26. □

Remark. In the course of writing this paper, it was brought to our attention that in a recently published paper, Durand and Rigo [4] give an alternative proof of the fact that polynomial and exponential growth cannot exists simultaneously. However, our approach is quite different, and it also gives the additional information of the polynomial degree.

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