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Approximating Traces*

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Abstract. In order to give a semantics to concurrent processes we need a model having some good mathematical properties. To this end we generalize (infinite) Mazurkiewicz traces by adding some alphabetical information concerning the possible continuations of a process. This allows to define an approximation order compatible with the composition. We obtain a prime algebraic and coherently complete domain where the compact elements are exactly the finite approximations of processes. The composition is shown to be monotone and $\sqcup$-continuous. We define a suitable metric which induces the Lawson topology and which yields a compact metric space being therefore complete. The finite approximations of processes form a dense and open subset and the composition is (uniformly) continuous. A preliminary version of this work appeared in [7].

1. Introduction

The aim of this paper is to provide a new semantic domain for concurrent systems having some good algebraic, topological, and order theoretic properties.

We want to be able to describe different stages of a behaviour, from its initialization to its termination. This will be abstracted through an approximation order allowing to deal with recursion, too. Each unfolding of the recursion corresponds to a new stage of the process. We thus obtain an increasing sequence of approximations of the actual behaviour. It is then natural to require that this increasing sequence of approximations converges to the semantics of the terminated behaviour. In the partial order setting, this limit can be defined by a least upper bound. Therefore we focus on some complete partial order (cpo) where we may apply fixed point theorems. Moreover, every element should be the limit of its finite approximations, hence we are led to consider a domain (algebraic cpo).

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Now, we want to be able to compute the semantics of a system in some modular way. The semantics of a composed process $P \odot Q$ should be the composition of the semantics of P and Q, and in order to give a semantics to a language including recursion, the operations should be continuous.

In the sequential case, a possible domain is given by the set of strings over some alphabet. One might be tempted to use the concatenation for the composition of processes and the prefix relation for the approximation relation. However, the concatenation of strings is clearly not monotone with respect to the prefix ordering since, e.g., a is a prefix of ab, but a^2 is no prefix of aba. The classical solution to this problem is to consider *streams* or strings together with a non-termination symbol $\perp$ (or equivalently a termination symbol $\top$ or $\surd$), see [3]. The domain is thus $\Sigma^*\perp \cup \Sigma^\infty$ and $u\perp$ means that u has already been executed and that termination of the process has not been observed yet. The concatenation is defined such that the non-termination symbol $\perp$ is right-absorbent: the composition of u and v is uv if $u \in \Sigma^*$ and is u otherwise. The approximation relation is defined by $u \sqsubseteq v$ if either $u = v$ or $u = v'\perp$ and $v = v'v''$. For instance,

$$1\perp \sqsubseteq a\perp \sqsubseteq ab\perp \sqsubseteq aba\perp \sqsubseteq aba.$$

It is easy to verify that the composition operation is monotone with respect to the approximation relation.

The situation is much more interesting when concurrency comes into play since some components may be terminated while others may still go on. As an easy example, consider a system composed of k fully independent processes. A terminated behaviour is a tuple $p = (u_1, \ldots, u_k)$ where $u_i \in \Sigma^\infty$ for all $1 \leq i \leq k$. In an approximation p' of p, some components may be terminated, say $i+1, \ldots, k$, while others may still go on. Thus, $p' = (u'_1\perp, \ldots, u'_i\perp, u_{i+1}, \ldots, u_k)$ where u'_j is a prefix of u_j for all $1 \leq j \leq i$. Now, even if the first process has not yet reached its global termination, another process $q = (v_1, \ldots, v_k)$ may be (partially) started according to the following "sequential" composition:

$$(u'_1\perp, \ldots, u'_i\perp, u_{i+1}, \ldots, u_k) \odot (v_1, \ldots, v_k) = (u'_1\perp, \ldots, u'_i\perp, u_{i+1}v_{i+1}, \ldots, u_kv_k)$$

Our model will be a generalization of this situation. Note that the approximation of a composed process will effectively use some informations concerning the second process. This is not the case for sequential processes: when the first process is not terminated, the approximation of a composed process makes no use of the second process.

Since we are interested in concurrent systems, we need a model with true concurrency, as opposed to interleaving models. We have chosen to extend Mazurkiewicz traces [20] where concurrency is expressed by some fixed independence relation. This model is close to the sequential calculus but it copes with basic phenomena of concurrency. See [9] for the state of the art (of the year 1995) or some other survey [1, 5, 8, 21, 23]. Note that several other generalizations of Mazurkiewicz traces have been considered before, but none of them have all desired properties. First, infinite real traces have been introduced [10, 12, 13, 18], see also [21]. Real traces can be seen as the ideal completion of finite traces with the prefix ordering. However the composition of real traces is partially defined only and it is not monotone with respect to the prefix order. To obtain a totally defined concatenation, complex traces were introduced as the topological completion of finite traces for a suitable metric [6]. But once again the composition is not monotone with respect to the prefix order and the poset properties of complex traces are not fully satisfactory [11].

The main problem with all these approaches is that they use the prefix ordering as the approximation relation and that they do not distinguish between terminated (finite) processes and finite approximations. Our model takes the opposite view by extending the idea of non-termination exposed above to traces and by considering an approximation relation which is in some sense orthogonal to the prefix ordering.

An α-trace describes an approximation of a process by a pair (t, A) where t is a Mazurkiewicz (possibly infinite) trace describing what has already been observed and A is the set of actions which may still occur. If A is the empty set this means global termination. (If t is infinite, then A is never empty.)

We define a better approximation mainly by letting t grow with actions specified in A and by possibly reducing uncertainty about the future by dropping letters from A. We then define a composition which allows to start the second process even if the first one has not yet reached its global termination.

We obtain a prime algebraic and coherently complete domain with the finite α-traces being the compact elements. The composition is monotone and $\sqcup$-continuous. We then define a metric and show that it induces the Lawson topology [19] on α-traces. Hence, we also have a complete and compact metric space with the set of finite α-traces being dense. Moreover, the composition is uniformly continuous. We also show how the main topological results can be obtained using projective limits.

One should note the difference with failure (or ready) semantics in process algebra like CSP or CCS [16, 22], where the behavior in the next step is specified. Here the finite alphabetical information A is given for the whole future. It should be noted that defining composition laws using dependencies from the alphabetical information about the future has been done in process algebra too. Let us refer to [17, 24, 25].

In the sequential case, α-traces are slightly more concrete than strings with non-termination. An α-trace does not merely express non-termination, it tells which actions may still occur. In order to have a more abstract model, we introduce δ-traces which differ from α-traces only by their information about the future: a δ-trace specifies in fact which actions may be started without waiting for the global termination of the current process. Since some elementary informations are missing, the resulting domain fails to be prime algebraic and distributive in general.

2. Preliminaries

In this section, we first recall some definitions concerning poset theory and some results on real traces being used throughout. For the results on real traces we refer to [13] or to [12] for a survey on infinite traces. For the notions concerning domain theory we refer to [2, 15].

2.1. Poset theory

Let $(Z, \leq)$ be a poset, i.e., a partially ordered set. A subset $Y \subseteq Z$ is *coherent* if for all $x, y \in Y$, there exists some $z \in Z$ such that $x \leq z$ and $y \leq z$. A non-empty subset $Y \subseteq Z$ is *directed* for all $x, y \in Y$ there exists some $z \in Y$ such that $x \leq z$ and $y \leq z$. A subset $Y \subseteq Z$ is *bounded above* if there exists $z \in Z$ such that $x \leq z$ for all $x \in Y$. The poset $(Z, \leq)$ is a *coherently complete partial order* (a *ccpo* for short) if every coherent subset of Z has a least upper bound in Z. Similarly, the poset $(Z, \leq)$ is a *directed complete partial order* (a *dcpo* for short) if every directed subset of Z has a least upper bound in Z. Finally, the poset $(Z, \leq)$ is a *bounded complete partial order* (a *bcpo* for short) if every bounded subset of Z has a least upper bound in Z. Note that a ccpo is indeed both a dcpo and a bcpo. Whenever they exist, the least upper bound and the greatest lower bound of a subset $Y \subseteq Z$ will be denoted by $\sqcup Y$ and $\sqcap Y$ respectively.

Let $(Z, \leq)$ be a ccpo. An element $x \in Z$ is *prime* if for all *coherent* subsets $Y \subseteq Z$, $x \leq \bigsqcup Y$ implies $x \leq y$ for some $y \in Y$. For all $z \in Z$, we denote by $P(z) = \{x \in Z \mid x \leq z \text{ and } x \text{ is prime}\}$ the set of prime elements of Z less than or equal to z. A ccpo $(Z, \leq)$ is *prime algebraic* if $z = \bigsqcup P(z)$ for all $z \in Z$.

Similarly, let $(Z, \leq)$ be a dcpo. An element $x \in Z$ is *compact* if for all *directed* subsets $Y \subseteq Z$, the inequality $x \leq \bigsqcup Y$ implies $x \leq y$ for some $y \in Y$. For $z \in Z$, we denote by $K(z) = \{x \in Z \mid x \leq z \text{ and } x \text{ is compact}\}$ the set of compact elements less than or equal to z. A dcpo $(Z, \leq)$ is *algebraic* if $K(z)$ is directed and $z = \bigsqcup K(z)$ for all $z \in Z$. Note that, if any coherent pair of elements of Z admits a least upper bound (in particular if $(Z, \leq)$ is a ccpo or a bcpo) then $K(z)$ is directed for all $z \in Z$. Finally, an algebraic dcpo is called a *domain*.

For a poset $(Z, \leq)$ and $z \in Z$ we use the following standard notations:

$$\downarrow\! z = \{y \in Z \mid y \leq z\} \quad \text{and} \quad \uparrow\! z = \{y \in Z \mid y \geq z\}.$$

Let $(X, \leq)$ and $(Y, \leq)$ be posets. A mapping $f : X \longrightarrow Y$ is *monotone*, if $x \leq y$ implies $f(x) \leq f(y)$ for all $x \in X$, $y \in Y$. Let $(X, \leq)$ and $(Y, \leq)$ be dcpos. A monotone mapping $f : X \longrightarrow Y$ is *continuous*, if $f(\bigsqcup Z) = \bigsqcup f(Z)$ for all non-empty directed sets $Z \subseteq X$.

2.2. Real traces

A dependence alphabet is a pair (Σ, D) where Σ is a finite set of actions, called the alphabet, and $D \subseteq \Sigma \times \Sigma$ is a reflexive and symmetric relation over Σ, called the dependence relation. Its complement $I = \Sigma \times \Sigma \setminus D$ is the independence relation. We will represent a dependence alphabet by an undirected graph over Σ where self-loops are omitted. For instance, the dependence alphabet (Σ, D) defined by $\Sigma = \{a, b, c\}$ and $D = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$ will be represented by the following graph:

$$(\Sigma, D) = a \;\text{——}\; b \qquad c.$$

For $A \subseteq \Sigma$, we denote by $I(A) = \{b \in \Sigma \mid (a, b) \in I \text{ for all } a \in A\}$ the set of letters independent from all letters of A and we denote by $D(A) = \{b \in \Sigma \mid (a, b) \in D \text{ for some } a \in A\}$ the set of letters dependent from some letter of A.

A *real trace* over the dependence alphabet (Σ, D) is (an isomorphism class of) a labeled directed acyclic graph $[V, E, \lambda]$, where the set V of vertices is at most countable, $E \subseteq V \times V$ is the set of edges, $\lambda : V \to \Sigma$ is the labeling function, and it holds

– edges are between dependent vertices: for all $p, q \in V$,

$$(\lambda(p), \lambda(q)) \in D \iff p = q \text{ or } (p,q) \in E \text{ or } (q,p) \in E$$

– vertices have finite pasts: for all $p \in V$, ${\downarrow}p = \{q \in V \mid (q,p) \in E^*\}$ is finite (where E^* denotes the reflexive and transitive closure of the edge relation E).

The set of real traces is denoted by $\mathbb{R}(\Sigma, D)$, or simply by $\mathbb{R}$. A finite trace is a real trace with a finite set of vertices. The set of finite traces is denoted by $\mathbb{M}(\Sigma, D)$, or simply by $\mathbb{M}$.

Let $t = [V, E, \lambda]$ be a real trace. The *alphabet* of t is the set of letters which occur in t: $\mathrm{alph}(t) = \lambda(V)$. The *alphabet at infinity* of t is the set of letters which occur infinitely often in t: $\mathrm{alphinf}(t) = \{a \in \Sigma \mid \lambda^{-1}(a) \text{ is infinite}\}$. The *length* of t is defined by $|V|$ and it is denoted by $|t|$. The number $|\lambda^{-1}(a)|$ of occurrences of a in t is denoted by $|t|_a$.

The concatenation of two real traces is their disjoint union together with new edges from the first trace to the second one. Formally, let $r = [V_1, E_1, \lambda_1]$ and $s = [V_2, E_2, \lambda_2]$ be real traces (we may assume that $V_1 \cap V_2 = \emptyset$). The concatenation of r and s is defined only if $\mathrm{alphinf}(r) \times \mathrm{alph}(s) \subseteq I$. Then we set $r \cdot s = [V, E, \lambda]$ where

$$\begin{array}{rcl} V &=& V_1 \mathbin{\dot\cup} V_2 \\ E &=& E_1 \mathbin{\dot\cup} E_2 \mathbin{\dot\cup} \{(p_1, p_2) \in V_1 \times V_2 \mid (\lambda_1(p_1), \lambda_2(p_2)) \in D\} \\ \lambda &=& \lambda_1 \mathbin{\dot\cup} \lambda_2 \end{array}$$

The concatenation of real traces is associative and the empty trace $1 = [\emptyset, \emptyset, \emptyset]$ is the unity. The concatenation of two finite traces is always defined and $(\mathbb{M}, \cdot, 1)$ is a monoid. The canonical morphism $\varphi : \Sigma^* \longrightarrow \mathbb{M}$ defined by $\varphi(a) = [\{x\}, \emptyset, x \mapsto a]$ is surjective. In fact, φ defines an isomorphism between the monoid of finite traces and the quotient of the free monoid Σ^* by the congruence generated by $\{(ab, ba) \mid (a,b) \in I\}$.

To underline the fact that the concatenation on real traces is only partially defined, we say that $(\mathbb{R}, \cdot, 1)$ is a *partial monoid*. This concatenation extends easily to an infinite product. Let $(r_i)_{i<\omega}$ be a sequence of real traces such that $\mathrm{alphinf}(r_i) \times \mathrm{alph}(r_j) \subseteq I$ for all $i < j$, then $\prod_{i<\omega} r_i \in \mathbb{R}$ is defined as the disjoint union of the graphs r_i with new arcs from vertices of r_i to vertices of r_j whenever $i < j$ and the vertices have dependent labels. The canonical morphism $\varphi : \Sigma^* \longrightarrow \mathbb{M}$ extends to a mapping $\varphi : \Sigma^\infty \longrightarrow \mathbb{R}$ which turns out to be surjective.

A trace $r \in \mathbb{R}$ is a prefix of a trace $t \in \mathbb{R}$, denoted by $r \leq t$, if $t = r \cdot s$ for some $s \in \mathbb{R}$. The partial monoid $(\mathbb{R}, \cdot, 1)$ is left cancellative: let $r, s, t \in \mathbb{R}$ be such that $r \cdot s$ and $r \cdot t$ are well-defined, if $r \cdot s = r \cdot t$ then $s = t$. Hence, if $r \leq t$, there exists a unique trace $s \in \mathbb{R}$ such that $t = r \cdot s$. This unique trace will be denoted by $r^{-1}t$. Note that $(\mathbb{R}, \cdot, 1)$ is right-cancellative by finite traces only.

The prefix relation is a partial order on real traces. The poset $(\mathbb{R}, \leq)$ is prime algebraic and coherently complete, in particular, it is a bounded complete domain. A real trace is compact if and only if it is a finite trace. A real trace r is prime if and only if it is finite and has exactly one maximal vertex.

The following result is a generalization of Levi's lemma [4] to real traces. (For a proof including infinite traces see [12, Prop. 11.2.10].)

Lemma 1. *Let x, y, z, t be real traces such that the products xy and zt are defined. Then we have $xy = zt$ if and only if there exist traces $r, u, v, s \in \mathbb{R}$ such that $x = ru$, $y = vs$, $z = rv$, $t = us$ and* $\mathrm{alph}(u) \times \mathrm{alph}(v) \subseteq I$.

Let $s = [V, E, \lambda] \in \mathbb{R}$ be a real trace and let $A \subseteq \Sigma$ be a subalphabet. We denote by $\mu_A(s)$ the maximal prefix of s being independent of A, i.e., the restriction of the graph s to the set $\{p \in V \mid \lambda(\downarrow p) \times A \subseteq I\}$. We denote the alphabet of the corresponding suffix by $\sigma_A(s) = \mathrm{alph}(\mu_A(s)^{-1}s)$. The proof of the following lemma is straightforward.

Lemma 2. *Let $s, t \in \mathbb{R}$ be real traces and let $A, B \subseteq \Sigma$ be subalphabets, then we have:*

$$\mu_A(st) = \mu_A(s)\mu_{A\cup\sigma_A(s)}(t) \tag{1}$$

$$\sigma_A(st) = \sigma_A(s) \cup \sigma_{A\cup\sigma_A(s)}(t) \tag{2}$$

$$\mu_{A\cup B}(s) = \mu_A(\mu_B(s)) \tag{3}$$

$$\sigma_{A\cup B}(s) = \sigma_A(\mu_B(s)) \cup \sigma_B(s) \tag{4}$$

3. Alpha-Traces

In this section, we introduce α-traces forming our new semantic model.

Definition 1. *Let (Σ, D) be a dependence alphabet. An α-trace is an ordered pair (r, A) where $r \in \mathbb{R}(\Sigma, D)$ is a real trace and $A \subseteq \Sigma$ is such that* $\mathrm{alphinf}(r) \subseteq A$. *The set of α-traces is denoted by $\mathbb{F}(\Sigma, D)$ or simply by $\mathbb{F}$.*

Intuitively, the first component of an α-trace represents what has already occurred whereas the second component contains the actions which still may occur in the future. For instance, let $(\Sigma, D) = a$ —— b —— c. The α-trace $x = (acbaacbc, \{a\})$ tells us that the process has already executed $acbaacbc$, but it may be not terminated, the only action that the process may still perform is a. Note that this gives us more informations than the non-termination symbol $\perp$. Here, we know that all occurrences of b and c have been performed while some a may still occur.

Let $x = (r, A)$ be an α-trace. The first component r of x is called its *real part* and is denoted by $\mathrm{Re}(x)$. The second component A of x is called its *imaginary part* and is denoted by $\mathrm{Im}(x)$. This imaginary part is not visible but it has influence on the composition of α-traces. The length of the α-trace x is the length of its real part and is denoted by $|x|$. Finally, the alphabet of the α-trace x is the set $\mathrm{alph}(x) = \mathrm{alph}(r) \cup A$.

Definition 2. *Let $x = (r, A)$ and $y = (s, B)$ be α-traces. We say that x is an approximation of y, denoted by $x \sqsubseteq y$, if $r \leq s$ and $B \cup \mathrm{alph}(r^{-1}s) \subseteq A$.*

Intuitively, x is an approximation of y if y contains more information about the actual process than x. In our setting, this is described by three different properties: first, the behaviour actually observed in x is a prefix of the behaviour actually observed in y, i.e., the real part r of x is a prefix of the real part s of y; second, all actions which occur in $r^{-1}s$ have to be non terminated in x, i.e., $\mathrm{alph}(r^{-1}s)$ is contained in the imaginary part of x; third, all actions which are non-terminated in y are also non terminated in x, i.e., the imaginary part of y is contained in the imaginary part of x. these three conditions yield precisely Definition 2.

For instance, let $(\Sigma, D) = a$ —— b —— c. Then,

$$(1,\Sigma) \sqsubseteq (acba,\Sigma) \sqsubseteq (acbaac,\{b,c\}) \sqsubseteq (acbaaccb,\{c\}) \sqsubseteq (acbaaccb,\emptyset)$$

and

$$(1,\Sigma) \sqsubseteq (acb,\Sigma) \sqsubseteq ((acb)^2,\Sigma) \sqsubseteq \cdots \sqsubseteq ((acb)^\omega,\Sigma)$$

The bottom element with respect to $\sqsubseteq$ is the α-trace $(1,\Sigma)$. It signifies that nothing has happened and everything is possible.

Proposition 1. *The set of α-traces $(\mathbb{F}(\Sigma,D),\sqsubseteq)$ form a poset (partially ordered set) with $(1,\Sigma)$ as least element.*

Proof. The approximation relation is reflexive and antisymmetric. Now assume that $(r,A) \sqsubseteq (s,B) \sqsubseteq (t,C)$. From $r \le s \le t$ we deduce $r \le t$ and $r^{-1}t = (r^{-1}s)(s^{-1}t)$. Hence $\mathrm{alph}(r^{-1}t) = \mathrm{alph}(r^{-1}s) \cup \mathrm{alph}(s^{-1}t) \subseteq A \cup B \subseteq A$. Finally, $C \subseteq B \subseteq A$ and we obtain $(r,A) \sqsubseteq (t,C)$. Thus, the approximation relation is a partial order on the set $\mathbb{F}$ of α-traces. The least element is clear by definition.

There is an embedding $\chi : \mathbb{R} \hookrightarrow \mathbb{F}$ from the set of real traces into the set of α-traces defined by $\chi(r) = (r, \mathrm{alphinf}(r))$. Therefore, we will identify a real trace $r \in \mathbb{R}$ with the corresponding α-trace $\chi(r) = (r, \mathrm{alphinf}(r))$, and we will write $\mathbb{M} \subseteq \mathbb{R} \subseteq \mathbb{F}$.

An α-trace is maximal for the approximation relation if and only if it is a real trace. Indeed, let $x = (r, \mathrm{alphinf}(r))$ be a real trace and assume that x is an approximation of the α-trace $y = (s,B)$. Since r is a prefix of s, it holds $\mathrm{alph}(r^{-1}s) \times \mathrm{alphinf}(r) \subseteq I$. Therefore $\mathrm{alph}(r^{-1}s) \subseteq \mathrm{Im}(x) = \mathrm{alphinf}(r)$ implies $\mathrm{alph}(r^{-1}s) = \emptyset$, whence $r = s$. Finally, the third condition of the approximation relation implies $\mathrm{alphinf}(r) = \mathrm{alphinf}(s) \subseteq B \subseteq \mathrm{alphinf}(r)$. Therefore, $x = y$ proving that all real traces are maximal for the approximation order. Conversely, assume that the α-trace $x = (r,A)$ is maximal. Clearly, x is an approximation of the real trace $y = (r, \mathrm{alphinf}(r))$. Hence, $x = y$ is a real trace.

We turn now to the definition of the concatenation (composition) of α-traces. This definition uses the maximal prefix $\mu_A(s)$ of a real trace s independent from a subalphabet A and the alphabet of the corresponding suffix $\sigma_A(s)$ which were introduced in Sect. 2.

Definition 3. *Let $x = (r,A)$ and $y = (s,B)$ be α-traces. The concatenation of x and y is defined by the following formula.*

$$x \cdot y = (r\mu_A(s), A \cup B \cup \sigma_A(s))$$

Let us explain the semantics of this concatenation. Assume that $x = (r,A)$ and $y = (s,B)$ are approximations of actual processes p and q. The composition of x and y should give us all possible informations concerning the composition of p and q. We know that r is a prefix of p and s is a prefix of q. Since the actions in A may still occur in p, only actions of q independent from A may be started. Hence, we know for sure that $r\mu_A(s)$ is a prefix of the composition of p and q. Finally, actions which may still be performed in the composition of p and q after $r\mu_A(s)$ has been executed are either actions which may still be performed in p

after r, i.e., actions from A, or actions which may still be performed in q after $\mu_A(s)$, i.e., actions from $B \cup \sigma_A(s)$.

For instance, let $(\Sigma, D) = a$ —— b —— c. Then (representing a trace by its lexicographical first word) we have:

$$\begin{array}{rcl}
(acbac, \{c\}) \cdot (aaccb, \emptyset) & = & (acbaaac, \{b, c\}) \\
(acbac, \Sigma) \cdot (aaccb, \emptyset) & = & (acbac, \Sigma) \\
(acbac, \emptyset) \cdot (aaccb, \{a\}) & = & (acbaaacccb, \{a\}) \\
(acbac, \{a\}) \cdot (aaccb, \{c\}) & = & (acbaccc, \Sigma)
\end{array}$$

Note that the bottom element $(1, \Sigma)$ for the approximation relation is right-absorbent with respect to the concatenation and we have $(r, A) \cdot (1, \Sigma) = (r, \Sigma)$ for all $(r, A) \in \mathbb{F}$.

Remark 1. Let us briefly explain how recursion can be handled by our approach. Consider a recursive process p of type $p = \mathrm{rec}\, x.\ t \cdot x$ where t is a finite trace. Its semantics should be given by some fixed point of the mapping $x \longmapsto (t, \emptyset) \cdot x$. The least fixed point of this mapping is the α-trace (t^ω, Σ) which blocks all future actions including actions independent of t. However from the syntax of the process p, we see that it will contain actions of $\mathrm{alph}(t)$, only. Following our rule that alphabetical information about the future should be taken into account, we start the iteration with $(1, \mathrm{alph}(t))$. Then we obtain a sequence:

$$\begin{array}{rclcl}
(1, \mathrm{alph}(t)) & \longmapsto & (t, \emptyset) \cdot (1, \mathrm{alph}(t)) & = & (t, \mathrm{alph}(t)) \\
& \longmapsto & (t, \emptyset) \cdot (t, \mathrm{alph}(t)) & = & (t^2, \mathrm{alph}(t)) \\
& \longmapsto & (t, \emptyset) \cdot (t^2, \mathrm{alph}(t)) & = & (t^3, \mathrm{alph}(t)) \\
& \longmapsto & \cdots & &
\end{array}$$

Thus the least fixed point being above $(1, \mathrm{alph}(t))$ is $(t^\omega, \mathrm{alph}(t))$. This is indeed the intended semantics of the recursion $p = \mathrm{rec}\, x.\ t \cdot x$.

Proposition 2. *The set $\mathbb{F}(\Sigma, D)$ is a monoid with the concatenation $\cdot$ as composition and the unit element $(1, \emptyset)$. Moreover, the embedding $\chi : \mathbb{R} \hookrightarrow \mathbb{F}$ from the set of real traces into the set of α-traces is a partial morphism.*

Proof. Clearly, $(1, \emptyset)$ is a unit element for the concatenation. Now, let $x = (r, A)$, $y = (s, B)$ and $z = (t, C)$ be α-traces. Using Equations 1–4 of Sect. 2, we deduce

$$\begin{array}{rcl}
\mathrm{Re}(x(yz)) & = & r\mu_A(s\mu_B(t)) \\
& = & r\mu_A(s)\mu_{A \cup \sigma_A(s)}(\mu_B(t)) \\
& = & r\mu_A(s)\mu_{A \cup B \cup \sigma_A(s)}(t) \\
& = & \mathrm{Re}((xy)z)
\end{array}$$

and

$$\begin{array}{rcl}
\mathrm{Im}(x(yz)) & = & A \cup (B \cup C \cup \sigma_B(t)) \cup \sigma_A(s\mu_B(t)) \\
& = & A \cup B \cup C \cup \sigma_B(t) \cup \sigma_A(s) \cup \sigma_{A \cup \sigma_A(s)}(\mu_B(t)) \\
& = & (A \cup B \cup \sigma_A(s)) \cup C \cup \sigma_{A \cup B \cup \sigma_A(s)}(t) \\
& = & \mathrm{Im}((xy)z)
\end{array}$$

Therefore, the concatenation is associative and the triple $(\mathbb{F}, \cdot, (1, \emptyset))$ is a monoid.

Finally, let $r, s \in \mathbb{R}$ be real traces such that $\operatorname{alphinf}(r) \times \operatorname{alph}(s) \subseteq I$. We have $\mu_{\operatorname{alphinf}(r)}(s) = s$. Hence, $\chi(r)\chi(s) = (rs, \operatorname{alphinf}(r) \cup \operatorname{alphinf}(s)) = \chi(rs)$.

Remark 2. Let $(\Sigma, D) = a \longrightarrow b \longrightarrow c$. A first attempt to define a concatenation of real traces might ask for the following laws: $a^\omega b^\omega = a^\omega$, $b^\omega c^\omega = b^\omega$, and $a^\omega c^\omega = (ac)^\omega$. However, then we have:

$$(a^\omega b^\omega)c^\omega = a^\omega c^\omega = (ac)^\omega \neq a^\omega = a^\omega b^\omega = a^\omega (b^\omega c^\omega) \ .$$

Since for obvious reasons (a and c are independent) we want to keep the law that $a^\omega c^\omega \neq a^\omega$, we have to change the law for $a^\omega b^\omega$ (and, of course, for $b^\omega c^\omega$) in order to have an associative operation. This is what happens in our calculus: The embedding of $\mathbb{R}$ into $\mathbb{F}$ defined above identifies x^ω with the α-trace $(x^\omega, \{x\})$ for $x \in \Sigma$. Hence $a^\omega b^\omega = (a^\omega, \{a\})(b^\omega, \{b\}) = (a^\omega, \{a, b\}) \neq a^\omega$. Here are some more lines of the associative calculus of α-traces:

$$\begin{aligned}
(a^\omega, \{a\})(b^\omega, \{b\})(c^\omega, \{c\}) &= (a^\omega, \Sigma) \\
(a^\omega, \{a\})(c^\omega, \{c\})(b^\omega, \{b\}) &= ((ac)^\omega, \Sigma) \\
(a^\omega, \{a\})((ac)^\omega, \{a, c\}) &= ((ac)^\omega, \{a, c\})
\end{aligned}$$

The monoid of α-traces is not cancellative. For all real traces $r \in \mathbb{R}$, the α-trace (r, Σ) is a left- zero. It is not right-cancellative either since, for instance, $a \cdot a^\omega = a^2 \cdot a^\omega$. Nevertheless, the following proposition holds.

Proposition 3. *The monoid $\mathbb{F}(\Sigma, D)$ is left-cancellative by finite real traces, i.e., for all $f \in \mathbb{M}(\Sigma, D), x, y \in \mathbb{F}(\Sigma, D)$ we have:*

$$f \cdot x = f \cdot y \Longrightarrow x = y$$

Proof. Let $f = (f, \emptyset) \in \mathbb{M}$ be a finite trace and let $x = (r, A)$ and $y = (s, B)$ be α-traces. Since $\mathbb{R}$ is left-cancellative by finite traces, we deduce from $(f \cdot r, A) = f \cdot x = f \cdot y = (f \cdot s, B)$ that $r = s$ and $A = B$, i.e., $x = y$.

Hence by the proposition above, if a finite trace $f \in \mathbb{M}$ is a prefix of an α-trace $x \in \mathbb{F}$, the corresponding suffix is well-defined and will be denoted by $f^{-1}x$.

Besides the approximation order, the prefix relation, which may be defined in any monoid, is also of interest. It will allow us to define an infinite product on α-traces.

Definition 4. *Let $x, y \in \mathbb{F}(\Sigma, D)$ be α-traces. Then x is a prefix of y, denoted by $x \leq y$, if $xz = y$ for some α-trace $z \in \mathbb{F}(\Sigma, D)$.*

Remark 3. Let $x = (r, A)$ and $y = (s, B)$ be α-traces. Then,

$$x \leq y \Longleftrightarrow r \leq s, A \subseteq B \text{ and } \operatorname{alph}(r^{-1}s) \subseteq I(A)$$

As in any monoid, the prefix relation is reflexive and transitive. From Remark 3, we deduce that the prefix relation on α-traces is also antisymmetric, whence is a partial order relation. The set of α-traces provided with the prefix relation is thus a poset with $1 = (1, \emptyset)$ as least element. Note that the prefix ordering $\leq$ and the approximation ordering $\sqsubseteq$ are disjoint in the following sense: If $(r, A) \leq (s, B)$ and $(r, A) \sqsubseteq (s, B)$ then we have $(r, A) = (s, B)$.

In order to define the infinite product on α-traces we first need to prove that every $\leq$-increasing sequence of α-traces has a $\leq$-least upper bound. This is a corollary of the following stronger result.

Proposition 4. *The poset $(\mathbb{F}(\Sigma, D), \leq)$ with the prefix ordering is a a coherently complete partial order. Moreover, if $X \subseteq \mathbb{F}(\Sigma, D)$ is an $\leq$-coherent set, then we have $\bigsqcup_{\leq} X = (t, C)$ where $t = \bigsqcup \mathrm{Re}(X)$ and $C = \mathrm{alphinf}(t) \cup (\bigcup_{x \in X} \mathrm{Im}(x))$.*

Proof. Let $X \subseteq \mathbb{F}$ be a coherent set. For all $x, y \in \mathbb{F}$, there exists $z \in \mathbb{F}$ such that $x \leq z$ and $y \leq z$, whence $\mathrm{Re}(x) \leq \mathrm{Re}(z)$ and $\mathrm{Re}(y) \leq \mathrm{Re}(z)$. Therefore, the set $\mathrm{Re}(X) \subseteq \mathbb{R}$ is coherent and $t = \bigsqcup \mathrm{Re}(X)$ exists. Let $C = \mathrm{alphinf}(t) \cup (\bigcup_{x \in X} \mathrm{Im}(x))$ and let $z = (t, C) \in \mathbb{F}$. We claim that $z = \bigsqcup_{\leq} X$.

Let $x = (r, A) \in X$. By definition, $r \leq t$ and $A \subseteq C$. In order to obtain $x \leq z$, it remains to show that $\mathrm{alph}(r^{-1}t) \subseteq I(A)$ (Remark 3). Let $a \in \mathrm{alph}(r^{-1}t)$. There exists $y = (s, B) \in X$ such that $|r|_a < |s|_a \leq |t|_a$. Since X is coherent, there exists $z' = (t', C') \in \mathbb{F}$ such that $x \leq z'$ and $y \leq z'$. Hence, $|s|_a \leq |t'|_a$ and $a \in \mathrm{alph}(r^{-1}t') \subseteq I(A)$. Therefore, z is an upper bound of X.

Let $y = (s, B) \in \mathbb{F}$ and assume that y is an upper bound for X. For all $x \in X$, $\mathrm{Re}(x) \leq \mathrm{Re}(y) = s$ and $\mathrm{Im}(x) \subseteq \mathrm{Im}(y) = B$. Hence, we obtain $t = \bigsqcup \mathrm{Re}(X) \leq s$ and $C = \mathrm{alphinf}(t) \cup (\bigcup_{x \in X} \mathrm{Im}(x)) \subseteq \mathrm{alphinf}(s) \cup B = B$. Since $t \leq s$, it holds $\mathrm{alph}(t^{-1}s) \subseteq I(\mathrm{alphinf}(t))$. On the other hand, for all $x \in X$, $\mathrm{Re}(x) \leq t \leq s$, whence $\mathrm{alph}(t^{-1}s) \subseteq \mathrm{alph}(\mathrm{Re}(x)^{-1}s) \subseteq I(\mathrm{Im}(x))$. Hence, $\mathrm{alph}(t^{-1}s) \subseteq \bigcap_{x \in X} I(\mathrm{Im}(x)) = I(\bigcup_{x \in X} \mathrm{Im}(x))$. Finally, $\mathrm{alph}(t^{-1}s) \subseteq I(\mathrm{alphinf}(t)) \cap I(\bigcup_{x \in X} \mathrm{Im}(x)) = I(C)$. Therefore, $z \leq y$ which concludes the proof.

Thanks to the previous proposition, we can define the ω-product on α-traces.

Definition 5. *Let $(x_i)_{i<\omega}$ be an ω-sequence of α-traces. The ω-product of $(x_i)_{i<\omega}$ is the $\leq$-least upper bound of the $\leq$-increasing sequence $(\prod_{j<i} x_j)_{i<\omega}$:*

$$\prod_{i<\omega} x_i = \bigsqcup_{i<\omega}^{\leq} \left(\prod_{j<i} x_j \right)$$

For instance, let $(\Sigma, D) = a \longrightarrow b \longrightarrow c \longrightarrow d$ and let $x_i = (a^i bd, \{d\})$ for all $i < \omega$. Then, $\prod_{i<\omega} x_i = (dbaba^2ba^3b\cdots, \{a, b, d\})$.

The monoid $\mathbb{M}(\Sigma, D)$ of finite traces is strictly embedded in the monoid of *finite* α-traces, denoted by $\mathbb{F}_f(\Sigma, D)$ or simply by $\mathbb{F}_f$. The alphabet Σ is a finite set of generators of $\mathbb{M}(\Sigma, D)$, hence, Σ cannot also generate the monoid $\mathbb{F}_f(\Sigma, D)$. In order to get a finite set of generators of $\mathbb{F}_f(\Sigma, D)$, we will use another copy of Σ.

Let $\xi : \mathcal{P}(\Sigma) \longrightarrow \mathbb{F}_f(\Sigma, D)$ defined by $\xi(A) = (1, A)$ for all $A \subseteq \Sigma$. Clearly, ξ is an injective morphism, hence defines an embedding of the power-set monoid

$(\mathcal{P}(\Sigma), \cup)$ into the monoid $\mathbb{F}_f(\Sigma, D)$ of finite α-traces. For $A \subseteq \Sigma$, we will also simply denote the α-trace $\xi(A)$ by A.

Clearly, a finite α-trace $x = (r, A) \in \mathbb{F}_f(\Sigma, D)$ is the concatenation of the finite trace r by the α-trace A:

$$\mathbb{F}_f(\Sigma, D) = \mathbb{M}(\Sigma, D) \cdot \mathcal{P}(\Sigma)$$

Let $\mathcal{S}(\Sigma) = \{\{a\} \mid a \in \Sigma\}$ be the set of singletons from Σ. Since $\mathbb{M}(\Sigma, D)$ is generated by Σ and $(\mathcal{P}(\Sigma), \cup)$ is generated by $\mathcal{S}(\Sigma)$ we deduce that $\Sigma \cup \mathcal{S}(\Sigma)$ is a finite set of generators of $\mathbb{F}_f(\Sigma, D)$.

In order to generate infinite α-traces, we will use the ω-product defined above. Let $x = (r, A) \in \mathbb{F}(\Sigma, D)$ be an arbitrary α-trace. As above, we have $x = r \cdot A$ therefore,

$$\mathbb{F}(\Sigma, D) = \mathbb{R}(\Sigma, D) \cdot \mathcal{P}(\Sigma)$$

Since any real trace is a finite product or an ω-product of letters from Σ, we deduce that $\Sigma \cup \mathcal{S}(\Sigma)$ is a finite set of generators of the set $\mathbb{F}(\Sigma, D)$ provided both with the concatenation and with the ω-product.

Note that, following Def. 5, we may also define γ-products for arbitrary ordinals γ. We restrict ourselves to ω-products for two reasons. First, ω-products are sufficient to generate all α-traces from $\Sigma \cup \mathcal{S}(\Sigma)$. Second, in an arbitrary γ-product, at most $\omega \cdot |\Sigma|$ factors are significant. Hence, every γ-product may be obtained by ω-products and finitely many concatenations.

The concatenation on finite words is not monotone with respect to the prefix order. The same holds for traces, and therefore for α-traces since the monoid of finite traces can be embedded. It is enough to consider $(a, b) \in D, a \neq b$. Then $a \leq ab$ while $a^2 \not\leq aba$. As another example, let $(\Sigma, D) = a$ —— b —— c. Then, $(a, \{c\})$ is a prefix of $(a, \Sigma) = (a, \{c\})(b, \{a\})$ while $(a, \{c\})^2 = (a^2, \{c\})$ is no prefix of $(a, \Sigma)^2 = (a, \Sigma)$.

It is just the approximation order which behaves well with respect to the concatenation.

Theorem 1. *The concatenation on α-traces is monotone for the approximation order: Let $x, x', y, y' \in \mathbb{F}(\Sigma, D)$ be α-traces. If $x \sqsubseteq x'$ and $y \sqsubseteq y'$ then we have $x \cdot x' \sqsubseteq y \cdot y'$.*

Proof. Let $x = (r, A)$, $x' = (r', A')$, $y = (s, B)$ and $y' = (s', B')$ be α-traces. Assume that $x \sqsubseteq x'$ and $y \sqsubseteq y'$, i.e., $r \leq r'$, $\mathrm{alph}(r^{-1}r') \cup A' \subseteq A$ and $s \leq s'$, $\mathrm{alph}(s^{-1}s') \cup B' \subseteq B$. We have:

$$\begin{aligned} x \cdot y &= (r\mu_A(s), A \cup B \cup \sigma_A(s)), \\ x' \cdot y' &= (r'\mu_{A'}(s'), A' \cup B' \cup \sigma_{A'}(s')). \end{aligned}$$

Since $s \leq s'$ and $A' \subseteq A$, it holds $\mu_A(s) \leq \mu_A(s') \leq \mu_{A'}(s')$. Hence,

$$r'\mu_{A'}(s') = r(r^{-1}r')\mu_A(s)(\mu_A(s)^{-1}\mu_{A'}(s')).$$

From $\mathrm{alph}(r^{-1}r') \subseteq A$, we deduce

$$r'\mu_{A'}(s') = r\mu_A(s)(r^{-1}r')(\mu_A(s)^{-1}\mu_{A'}(s')).$$

Therefore, $\mathrm{Re}(xy) = r\mu_A(s) \leq r'\mu_{A'}(s') = \mathrm{Re}(x'y')$ and

$$
\begin{aligned}
&\mathrm{Im}(x'y') \cup \mathrm{alph}(\mathrm{Re}(xy)^{-1}\,\mathrm{Re}(x'y')) \\
&\qquad = A' \cup B' \cup \sigma_{A'}(s') \cup \mathrm{alph}((r^{-1}r')(\mu_A(s)^{-1}\mu_{A'}(s'))) \\
&\qquad = A' \cup \mathrm{alph}(r^{-1}r') \cup B' \cup \mathrm{alph}((\mu_A(s)^{-1}\mu_{A'}(s'))(\mu_{A'}(s')^{-1}s')) \\
&\qquad = A' \cup \mathrm{alph}(r^{-1}r') \cup B' \cup \mathrm{alph}(\mu_A(s)^{-1}s') \\
&\qquad = A' \cup \mathrm{alph}(r^{-1}r') \cup B' \cup \mathrm{alph}((\mu_A(s)^{-1}s)(s^{-1}s')) \\
&\qquad = A' \cup \mathrm{alph}(r^{-1}r') \cup B' \cup \mathrm{alph}(s^{-1}s') \cup \sigma_A(s) \\
&\qquad \subseteq A \cup B \cup \sigma_A(s) = \mathrm{Im}(xy).
\end{aligned}
$$

In order to prove that the ω-product is also monotone with respect to the approximation order, we first establish that the $\leq$-least upper bound is $\sqsubseteq$-monotone.

Let $X, Y \subseteq \mathbb{F}$ be sets of α-traces. We say that X approximates Y, denoted by $X \sqsubseteq Y$, if for all $x \in X$ there exists $y \in Y$ such that $x \sqsubseteq y$ and for all $y \in Y$ there exists $x \in X$ such that $x \sqsubseteq y$.

Lemma 3. *Let $X, Y \subseteq \mathbb{F}(\Sigma, D)$ be $\leq$-coherent sets of α-traces such that X approximates Y. Then, $\bigsqcup_{\leq} X \sqsubseteq \bigsqcup_{\leq} Y$.*

Proof. Let $x = \bigsqcup_{\leq} X = (r, A)$ and let $y = \bigsqcup_{\leq} Y = (s, B)$. For all $x' \in X$ there exists $y' \in Y$ such that $x' \sqsubseteq y'$, whence $\mathrm{Re}(x') \leq \mathrm{Re}(y') \leq s$. Therefore, $r = \bigsqcup \mathrm{Re}(X) \leq s = \bigsqcup \mathrm{Re}(Y)$.

We will now prove that $B \cup \mathrm{alph}(r^{-1}s) \subseteq A$. Recall that by Proposition 4, $A = \mathrm{alphinf}(r) \cup (\bigcup_{x' \in X} \mathrm{Im}(x'))$ and $B = \mathrm{alphinf}(s) \cup (\bigcup_{y' \in Y} \mathrm{Im}(y'))$. First, let $y' \in Y$. There exists $x' \in X$ such that $x' \sqsubseteq y'$. Hence, $\mathrm{Im}(y') \subseteq \mathrm{Im}(x') \subseteq A$. Second, let $a \in \mathrm{alph}(r^{-1}s)$. There exists $y' = (s', B') \in Y$ such that $|r|_a < |s'|_a \leq |s|_a$. There exists $x' = (r', A') \in X$ such that $x' \sqsubseteq y'$. Since $x' \leq x$, we have $r' \leq r$ and $|r'|_a \leq |r|_a < |s'|_a$. Therefore, $a \in \mathrm{alph}(r'^{-1}s') \subseteq A' = \mathrm{Im}(x') \subseteq A$. Third, $\mathrm{alphinf}(s) \setminus \mathrm{alphinf}(r) \subseteq \mathrm{alph}(r^{-1}s)$ and we obtain $B \cup \mathrm{alph}(r^{-1}s) \subseteq A$.

Proposition 5. *The ω-product on α-traces is $\sqsubseteq$-monotone: Let $(x_i)_{i<\omega}$ and $(y_i)_{i<\omega}$ be sequences of α-traces such that $x_i \sqsubseteq y_i$ for all $i < \omega$. Then,*

$$\prod_{i<\omega} x_i \sqsubseteq \prod_{i<\omega} y_i.$$

Proof. From Theorem 1, we deduce that $\prod_{j<i} x_j \sqsubseteq \prod_{j<i} y_j$ for all $i < \omega$. Therefore, $X = \{\prod_{j<i} x_j \mid i < \omega\}$ approximates $Y = \{\prod_{j<i} y_j \mid i < \omega\}$. It follows from Lemma 3 that

$$\prod_{i<\omega} x_i = \bigsqcup_{\leq} X \sqsubseteq \bigsqcup_{\leq} Y = \prod_{i<\omega} y_i.$$

4. Poset Properties

In this section, we will show that α-traces provide a good semantic domain. We first investigate completeness properties (c.f. Sect. 2.1) of the poset $(\mathbb{F}, \sqsubseteq)$. We show that $(\mathbb{F}, \sqsubseteq)$ is a coherently complete partial order (ccpo), whence a directed complete partial order (dcpo) and a bounded complete partial order (bcpo). Then we show that the compact elements of $(\mathbb{F}, \sqsubseteq)$ are the finite α-traces and that the poset is algebraic, whence a domain. Moreover, it is prime algebraic and we characterize the prime α-traces. Finally we show that the concatenation of α-traces is $\sqsubseteq$-continuous.

Theorem 2. *The poset $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ is a coherently complete partial order. Moreover, if $X \subseteq \mathbb{F}(\Sigma, D)$ is a coherent set of α-traces, then we have:*

$$\bigsqcup X = (\bigsqcup \mathrm{Re}(X), \bigcap_{x\in X} \mathrm{Im}(x)).$$

Proof. Let $X \subseteq \mathbb{F}$ be $\sqsubseteq$-coherent. For all $x, y \in X$, there exists $z \in \mathbb{F}$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$, whence $\mathrm{Re}(x) \leq \mathrm{Re}(z)$ and $\mathrm{Re}(y) \leq \mathrm{Re}(z)$. Therefore, the set $\mathrm{Re}(X) \subseteq \mathbb{R}$ is $\leq$-coherent and admits a $\leq$-least upper bound (see Sect. 2.2). Let $t = \bigsqcup \mathrm{Re}(X)$ and let $C = \bigcap_{x\in X} \mathrm{Im}(x)$.

Claim. For all $x = (r, A) \in X$, it holds $\mathrm{alph}(r^{-1}t) \subseteq A$.

Indeed, let $a \in \mathrm{alph}(r^{-1}t)$. There exists $y = (s, B) \in X$ such that $|r|_a < |s|_a \leq |t|_a$. Since X is coherent, there exists $z' = (t', C') \in \mathbb{F}$ such that $x \sqsubseteq z'$ and $y \sqsubseteq z'$. Hence, $|r|_a < |s|_a \leq |t'|_a$ and $a \in \mathrm{alph}(r^{-1}t') \subseteq A$.

The first consequence of this claim is that $z = (t, C)$ is an α-trace. Indeed, for all $x = (r, A) \in X$, we have $\mathrm{alphinf}(t) \subseteq \mathrm{alphinf}(r) \cup \mathrm{alph}(r^{-1}t) \subseteq A$. Therefore, $\mathrm{alphinf}(t) \subseteq C$.

The second consequence is that z is an upper bound of X.

Now let $y = (s, B) \in \mathbb{F}$ be an upper bound of X. Clearly, $t = \bigsqcup \mathrm{Re}(X) \leq s$ and $B \subseteq \bigcap_{x\in X} \mathrm{Im}(x) = C$. For all $x = (r, A) \in X$, we have $r \leq t \leq s$. Hence, $\mathrm{alph}(t^{-1}s) \subseteq \mathrm{alph}(r^{-1}s) \subseteq A$. Therefore, $\mathrm{alph}(t^{-1}s) \subseteq \bigcap_{x\in X} \mathrm{Im}(x) = C$ and $z \sqsubseteq y$.

It follows from Theorem 2 that the poset $(\mathbb{F}, \sqsubseteq)$ is a dcpo and a bcpo and that every non-empty set of α-traces admits a greatest lower bound. The following corollary gives, in addition, the expression of the greatest lower bound of a non-empty set of α-traces.

Corollary 1. *Let $Y \subseteq \mathbb{F}(\Sigma, D)$ be a non-empty set of α-traces. Then we have:*

$$\sqcap Y \quad = \quad (\sqcap \mathrm{Re}(Y), \cup\{\mathrm{Im}(y) \cup \mathrm{alph}(r^{-1}\,\mathrm{Re}(y)) \mid y \in Y\}).$$

Proof. Let $(r, A) = (\sqcap \mathrm{Re}(Y), \cup\{\mathrm{Im}(y) \cup \mathrm{alph}(r^{-1}\,\mathrm{Re}(y)) \mid y \in Y\})$ and $y \in Y$. We have $\mathrm{alphinf}(r) \subseteq \mathrm{alphinf}(\mathrm{Re}(y)) \subseteq \mathrm{Im}(y) \subseteq A$. Hence, $x = (r, A)$ is an α-trace being, by definition of r and A, a lower bound of Y.

Now, let $x' = (r', A')$ be a lower bound of Y. For all $y \in Y$, we have $x' \sqsubseteq y$, i.e., $r' \leq \mathrm{Re}(y)$ and $\mathrm{Im}(y) \cup \mathrm{alph}(r'^{-1}\,\mathrm{Re}(y)) \subseteq A'$. Hence, $r' \leq \sqcap \mathrm{Re}(Y) = r$ and $\bigcup_{y\in Y}(\mathrm{Im}(y) \cup \mathrm{alph}(r'^{-1}\,\mathrm{Re}(y))) \subseteq A'$. Since $r'^{-1}\,\mathrm{Re}(y) = (r'^{-1}r)(r^{-1}\,\mathrm{Re}(y))$, it follows that

$$\begin{aligned} A \cup \mathrm{alph}(r'^{-1}r) &= \bigcup_{y\in Y} (\mathrm{Im}(y) \cup \mathrm{alph}(r^{-1}\,\mathrm{Re}(y)) \cup \mathrm{alph}(r'^{-1}r)) \\ &= \bigcup_{y\in Y} (\mathrm{Im}(y) \cup \mathrm{alph}(r'^{-1}\,\mathrm{Re}(y))) \\ &\subseteq A'. \end{aligned}$$

Therefore, $x' \sqsubseteq x$ which concludes the proof.

We turn now to the domain properties of $(\mathbb{F}, \sqsubseteq)$. To this purpose, we first characterize compact α-traces.

Theorem 3. *The following properties hold.*

1. *An α-trace is compact if and only if it is finite.*
2. *The poset $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ is algebraic.*

Proof. 1. Let $y = (s, B) \in \mathbb{F}_f$ be a finite α-trace and let $X \subseteq \mathbb{F}$ be a directed set such that $y \sqsubseteq \bigsqcup X$. The set $\mathrm{Re}(X)$ is $\leq$-directed and $s \leq \bigsqcup \mathrm{Re}(X) = \mathrm{Re}(\bigsqcup X)$. Since $s \in \mathbb{M}$ is a finite trace, s is a compact element of the dcpo $(\mathbb{R}, \leq)$ (see Sect. 2.2). Hence, there exists $x_0 \in X$ such that $s \leq \mathrm{Re}(x_0)$. Now, for all $a \notin \mathrm{Im}(\bigsqcup X) = \bigcap_{x \in X} \mathrm{Im}(x)$, there exists $x_a \in X$ such that $a \notin \mathrm{Im}(x_a)$. Since X is directed, there exists $x \in X$ which is an upper bound of x_0 and $(x_a)_{a \notin \mathrm{Im}(\bigsqcup X)}$. We will show that $y \sqsubseteq x$. First, $\mathrm{Re}(y) = s \leq \mathrm{Re}(x_0) \leq \mathrm{Re}(x)$. Second,

$$\mathrm{Im}(x) \subseteq \bigcap_{a \notin \mathrm{Im}(\bigsqcup X)} \mathrm{Im}(x_a) \subseteq \mathrm{Im}(\bigsqcup X) \subseteq \mathrm{Im}(y).$$

Third, $\mathrm{alph}(\mathrm{Re}(y)^{-1}\,\mathrm{Re}(x)) \subseteq \mathrm{alph}(\mathrm{Re}(y)^{-1}\,\mathrm{Re}(\bigsqcup X)) \subseteq \mathrm{Im}(y)$. Therefore, $y \sqsubseteq x$ which proves that y is compact.

In order to prove the converse and the second part of the theorem, we will use the following claim.

Claim. Let $y = (s, B) \in \mathbb{F}$ be an arbitrary α-trace, then $X = {\downarrow} y \cap \mathbb{F}_f$ is directed and $\bigsqcup X = y$.

Let (r', A') and (r'', A'') be finite α-traces bounded above by y. Since r' and r'' are bounded above by s, the least upper bound $r = r' \vee r''$ exists, is finite and bounded above by s. From $r' \leq r \leq s$, we deduce that $r'^{-1}s = (r'^{-1}r)(r^{-1}s)$. Since $(r', A') \sqsubseteq y$, it follows that $(r', A') \sqsubseteq (r', B \cup \mathrm{alph}(r'^{-1}s)) \sqsubseteq (r, B \cup \mathrm{alph}(r^{-1}s))$. Similarly, $(r'', A'') \sqsubseteq (r, B \cup \mathrm{alph}(r^{-1}s))$ which proves that X is directed.

Note that $\mathrm{Re}(X)$ is the set of compact real traces lower than s. Since $\mathbb{R}$ is algebraic, it follows $s = \bigsqcup \mathrm{Re}(X) = \mathrm{Re}(\bigsqcup X)$. Let $r \in \mathbb{M}$ be such that $r \leq s$ and $\mathrm{alph}(r^{-1}s) = \mathrm{alphinf}(s)$. Since $\mathrm{alphinf}(s) \subseteq B$, we deduce that $(r, B \cup \mathrm{alph}(r^{-1}s)) = (r, B) \in X$. Therefore, $\mathrm{Im}(\bigsqcup X) = B$ which concludes the proof of the claim.

The first consequence of this claim is that a compact α-trace must be finite. Indeed, if $y = (s, B) \in \mathbb{F} \setminus \mathbb{F}_f$ is an infinite α-trace, then $y = \bigsqcup X$ but, being infinite, y is not lower than any member of $X \subseteq \mathbb{F}_f$.

2. By the first part we deduce that for all α-traces $y \in \mathbb{F}$, the set $K(y)$ of compact α-traces lower than y is precisely the set ${\downarrow} y \cap \mathbb{F}_f$. Therefore, the claim states in fact that the poset $(\mathbb{F}, \sqsubseteq)$ is algebraic.

By Theorems 2 and 3 we obtain.

Corollary 2. *The poset $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ is a domain, i.e., an algebraic dcpo.*

We address now the more complex problem of prime algebraicity of the poset $(\mathbb{F}, \sqsubseteq)$. As above, we will first characterize the prime α-traces. We introduce a notation. Let $r \in \mathbb{R}$ be a real trace. We denote by $\max(r)$ the set of letters which occur at maximal vertices of the graph r. For instance, let $(\Sigma, D) = a \text{---} b \text{---} c$. Then, $\max(acb) = \{b\}$, $\max(acbac) = \{a, c\}$ and $\max((acb)^\omega) = \emptyset$. Recall that a non-empty real trace r is prime if and only if it is finite and has exactly one maximal vertex ($|\max(r)| = 1$).

Proposition 6. *An α-trace $x = (r, A)$ is prime if and only if either $r = 1$ and $|\Sigma \setminus A| = 1$ or r is a prime real trace and $A \cup \max(r) = \Sigma$.*

Proof. The only if part is a rephrasing of the following necessary condition:

1. If $(r, A) \in \mathbb{F}$ is prime, then either r is a prime real trace or $r = 1$.
2. If $(r, A) \in \mathbb{F}$ is prime, then $|\Sigma \setminus A| \leq 1$.
3. If $(r, A) \in \mathbb{F}$ is prime and r is a prime real trace, then $A \cup \max(r) = \Sigma$.

Indeed, assume that $r = sa = tb$ for some $s, t \in \mathbb{M}$, $a, b \in \Sigma$, $(a, b) \in I$. Then the α-trace (r, A) is not prime since $(r, A) = (s, A \cup \{a\}) \vee (t, A \cup \{b\})$ but $(r, A) \not\sqsubseteq (s, A \cup \{a\})$ and $(r, A) \not\sqsubseteq (t, A \cup \{b\})$. Since a prime α-trace is also compact, its real part must be finite and the first point follows.

Now, assume that $a, b \in \Sigma \setminus A$ with $a \neq b$. It holds $(r, A) = (r, A \cup \{a\}) \vee (r, A \cup \{b\})$. Hence, (r, A) is not prime which proves the second assertion.

Finally, assume that there exists a letter $b \in \Sigma \setminus (A \cup \{a\})$ where $a = \max(r)$. We may write $r = sa$ and we see that (r, A) cannot be prime since we have $(r, A) = (s, A \cup \{a\}) \vee (r, A \cup \{b\})$. Hence the last condition.

Conversely, assume that $x = (r, A) \in \mathbb{F}$ satisfies the conditions of the proposition. Let $Y \subseteq \mathbb{F}$ be a coherent set such that $x \sqsubseteq \bigsqcup Y = z = (t, C)$. Note that since $(1, \Sigma)$ does not satisfies the conditions of the proposition, the set Y must be nonempty.

First, assume that $A = \Sigma$. We have $r \leq t = \bigsqcup \mathrm{Re}(Y)$. Since $r = 1$ or r is a prime real trace, there exists $y \in Y$ such that $r \leq \mathrm{Re}(y)$. It follows that $x = (r, \Sigma) \sqsubseteq y$ which proves that x is prime.

Second, assume that $\Sigma \setminus A = \{a\}$. Since $C = \bigcap_{y \in Y} \mathrm{Im}(y) \subseteq A$, there exists $y = (s, B) \in Y$ such that $a \notin B$, i.e., $B \subseteq A$. We claim that $x \sqsubseteq y$ which concludes the proof of the proposition. Indeed, assume that $r \not\leq s$. Then, r is prime and $\max(r) = \{a\}$. Since $x \sqsubseteq z$ and $y \sqsubseteq z$, it follows $a \in \mathrm{alph}(s^{-1}t) \subseteq B$ which forms a contradiction. Therefore, $r \leq s$ and from $x \sqsubseteq z$ we deduce that $\mathrm{alph}(r^{-1}s) \subseteq \mathrm{alph}(r^{-1}t) \subseteq A$. Thus, the claim $x \sqsubseteq y$.

Thanks to this characterization of prime α-traces, we will now prove:

Theorem 4. *The coherently complete domain $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ is prime algebraic.*

Proof. Let $x = (r, A) \in \mathbb{F}$ and let $P(x) = \{y \in \mathbb{F} \mid y \sqsubseteq x \text{ and } y \text{ is prime}\}$. Let $Y_1 = \{(s, \Sigma) \mid s \leq r \text{ and } s \text{ is prime}\}$. By Proposition 6, we have $Y_1 \subseteq P(x)$. Since $\mathbb{R}$ is prime algebraic, we deduce that $r = \bigsqcup \mathrm{Re}(Y_1) \leq \bigsqcup \mathrm{Re}(P(x)) \leq r$. From Theorem 2, we obtain $\mathrm{Re}(\bigsqcup P(x)) = \bigsqcup \mathrm{Re}(P(x)) = r$.

For all $a \in \Sigma \setminus A$, let $s_a \in \mathbb{R}$ be the least prefix of r such that $|s_a|_a = |r|_a$. Since $\mathrm{alphinf}(r) \subseteq A$, it holds $|r|_a < \infty$ and s_a is a finite real trace. Either $s_a = 1$, or s_a is a prime real trace with $\max(s_a) = \{a\}$. Let $y_a = (s_a, \Sigma \setminus \{a\}) \in \mathbb{F}$. By Proposition 6, the α-trace y_a is prime. Moreover, $y_a \sqsubseteq x$ and we obtain $Y_2 = \{y_a \mid a \in \Sigma \setminus A\} \subseteq P(x)$. Using Theorem 2, we deduce $A \subseteq \mathrm{Im}(\bigsqcup P(x)) \subseteq \bigcap_{a \notin A} \mathrm{Im}(y_a) = A$. Therefore, as desired $x = \bigsqcup P(x)$.

We conclude this section by showing that the concatenation on α-traces is continuous with respect to the approximation order. Note that, if $X, Y \subseteq \mathbb{F}$ are coherent sets then it follows from Theorem 1 that $X \cdot Y$ is coherent and that $\bigsqcup(X \cdot Y) \sqsubseteq (\bigsqcup X) \cdot (\bigsqcup Y)$. This inequality is strict in general, as shown by the following example. Let $(\Sigma, D) = a$ —— b —— c, let $X = \{(1,\{a\}),(1,\{c\})\}$ and let $Y = \{(b,\emptyset)\}$. Then, $X \cdot Y = \{(1,\{a,b\}),(1,\{b,c\})\}$ and $\bigsqcup(X \cdot Y) = (1,\{b\})$ while $(\bigsqcup X) \cdot (\bigsqcup Y) = (1,\emptyset)(b,\emptyset) = (b,\emptyset) \neq \bigsqcup(X \cdot Y)$.

Proposition 7. *The concatenation on $\mathbb{F}(\Sigma, D)$ is continuous with respect to the approximation order: Let $X, Y \subseteq \mathbb{F}(\Sigma, D)$ be directed sets, then $X \cdot Y$ is directed and $\bigsqcup(X \cdot Y) = (\bigsqcup X) \cdot (\bigsqcup Y)$.*

Proof. Let $X, Y \subseteq \mathbb{F}(\Sigma, D)$ be directed. As above, it follows directly from Theorem 1 that $X \cdot Y$ is directed and that $\bigsqcup(X \cdot Y) \sqsubseteq (\bigsqcup X) \cdot (\bigsqcup Y)$. Since $\mathbb{F}$ is algebraic (Theorem 3), the converse inequality follows directly from the claim: $z \sqsubseteq \bigsqcup(X \cdot Y)$ for all compact z such that $z \sqsubseteq (\bigsqcup X) \cdot (\bigsqcup Y)$.

Let $x = \bigsqcup X = (r, A)$, $y = \bigsqcup Y = (s, B)$ and $z = (t, C) \sqsubseteq x \cdot y$ be compact. We have $t \leq r\mu_A(s)$ and $A \cup B \cup \sigma_A(s) \cup \mathrm{alph}(t^{-1} r\mu_A(s)) \subseteq C$. Using the generalization of Levi's lemma to real traces (see Sect. 2.2), we deduce that $t = r's'$ where $r' \leq r$, $s' \leq \mu_A(s)$ and $\mathrm{alph}(s') \times \mathrm{alph}(r'^{-1}r) \subseteq I$. Note that this implies $t^{-1}r\mu_A(s) = (r'^{-1}r)(s'^{-1}\mu_A(s))$ and therefore $\sigma_A(s) \cup \mathrm{alph}(t^{-1}r\mu_A(s)) = \mathrm{alph}(r'^{-1}r) \cup \mathrm{alph}(s'^{-1}s)$.

Let $x' = (r', A \cup \mathrm{alph}(r'^{-1}r))$ and $y' = (s', C)$. We have $x' \sqsubseteq x$ and $y' \sqsubseteq y$. Since x' and y' are compact, it follows that $x' \sqsubseteq x''$ and $y' \sqsubseteq y''$ for some $x'' \in X$ and $y'' \in Y$. Now, $x' \cdot y' = (r's', A \cup \mathrm{alph}(r'^{-1}r) \cup C) = (t, C) = z$. Using Theorem 1, we obtain $z = x' \cdot y' \sqsubseteq x'' \cdot y'' \in X \cdot Y$ which concludes the proof of the claim.

Let us add another remark. Recall that the partial monoid of real traces $(\mathbb{R}, \cdot, 1)$ is embedded in the monoid of *alpha*-traces $(\mathbb{F}, \cdot, (1, \emptyset))$ by the partial morphism $\chi : r \mapsto (r, \mathrm{alphinf}(r))$. In fact, the poset $(\mathbb{R}, \leq)$ may also be embedded in the poset $(\mathbb{F}, \sqsubseteq)$ but the embedding is quite different. This time, we associate with a real trace $r \in \mathbb{R}$ the α-trace $\eta(r) = (r, \Sigma)$. Clearly, for all real traces $r, s \in \mathbb{R}$, we have $r \leq s$ if and only if $\eta(r) \sqsubseteq \eta(s)$. In other words, η is a poset morphism which embeds the poset $(\mathbb{R}, \leq)$ into the poset $(\mathbb{F}, \sqsubseteq)$. Note that the poset morphism η is continuous. In fact, if $R \subseteq \mathbb{R}$ is $\leq$-coherent then $\eta(R)$ is $\sqsubseteq$-coherent and $\eta(\bigsqcup R) = \bigsqcup \eta(R)$. Moreover, a real trace $r \in \mathbb{R}$ is $\leq$-compact ($\leq$-prime resp.) if and only if the α-trace $\eta(r)$ is $\sqsubseteq$-compact ($\sqsubseteq$-prime resp.).

5. Topology

In this section, we introduce a natural ultra-metric d on α-traces such that the topology being induced by this metric is precisely the Lawson topology [19] associated with the approximation order. Using the results of Sect. 4, we deduce that the metric space $(\mathbb{F}, d)$ is compact (and hence complete). Moreover, the set $\mathbb{F}_f$ of finite α-traces is discrete and dense in $(\mathbb{F}, d)$. Then, we prove that the concatenation is uniformly continuous. Finally, we show that the main topological properties of α-traces (compactness, concatenation uniformly continuous) can be obtained using projective limits.

5.1. A natural ultra-metric and the Lawson topology

In order to define the ultra metric, we first introduce the n^{th} approximation of a real trace and that of an α-trace.

Definition 6. *Let $n \in \mathbb{N}$.*

1. *Let $r \in \mathbb{R}(\Sigma, D)$, then the n^{th} approximation of r is the finite trace*
$$r[n] = \bigsqcup\{p \in \mathbb{M}(\Sigma, D) \mid p \leq r \text{ and } |p| \leq n\}.$$
Moreover, we denote by $r[-n] = r[n]^{-1}r$ the corresponding suffix.
2. *Let $x \in \mathbb{F}(\Sigma, D)$, then the n^{th} approximation of x is the finite α-trace*
$$x[n] = \bigsqcup\{y \in \mathbb{F}_f(\Sigma, D) \mid y \sqsubseteq x \text{ and } |y| \leq n\}.$$

Clearly, if $r \in \mathbb{R}$ is a real trace, then $(r[n])_{n\geq 0}$ is an $\leq$-increasing sequence bounded above by r. Hence, $\bigsqcup_{n\geq 0} r[n] \sqsubseteq r$. Conversely, let $K(r) \subseteq \mathbb{M}$ be the set of compact real traces lower than r. For all $k \in K(r)$, we have $k \leq r[|k|]$. Hence, $\bigsqcup K(r) \sqsubseteq \bigsqcup_{n\geq 0} r[n]$. Since $(\mathbb{R}, \leq)$ is algebraic we deduce that $r = \bigsqcup_{n\geq 0} r[n]$.

Similarly, for all α-traces $x \in \mathbb{F}$, the sequence $(x[n])_{n\geq 0}$ is $\sqsubseteq$-increasing and we have $x = \bigsqcup_{n\geq 0} x[n]$.

Proposition 8. *Let $x = (r, A) \in \mathbb{F}(\Sigma, D)$ be an α-trace and let $n \in \mathbb{N}$. Then,*
$$x[n] = (r[n], A \cup \operatorname{alph}(r[-n]))$$

Proof. Let $Y = \{y \in \mathbb{F} \mid y \sqsubseteq x \text{ and } |y| \leq n\}$. Let $R = \{p \in \mathbb{M} \mid p \leq r \text{ and } |p| \leq n\}$. Clearly, $\operatorname{Re}(Y) \subseteq R$. Conversely, let $p \in R$, then $y = (p, A \cup \operatorname{alph}(p^{-1}r)) \in Y$. Hence, $R = \operatorname{Re}(Y)$ and $\operatorname{Re}(x[n]) = \operatorname{Re}(\bigsqcup Y) = \bigsqcup \operatorname{Re}(Y) = \bigsqcup R = r[n]$.

Since $x[n] \sqsubseteq x$, we have $A \cup \operatorname{alph}(r[-n]) \subseteq \operatorname{Im}(x[n])$. Conversely, we have seen above that $\{(p, A \cup \operatorname{alph}(p^{-1}r)) \mid p \in R\} \subseteq Y$. Therefore,
$$\begin{aligned}\operatorname{Im}(x[n]) &= \operatorname{Im}(\bigsqcup Y) = \bigcap_{y\in Y} \operatorname{Im}(y) \\ &\subseteq \bigcap_{p\in R} (A \cup \operatorname{alph}(p^{-1}r)) = A \cup \left(\bigcap_{p\in R} \operatorname{alph}(p^{-1}r)\right).\end{aligned}$$
Hence, the result follows from the claim:

Claim.
$$\operatorname{alph}(r[-n]) = \bigcap_{p\in R} \operatorname{alph}(p^{-1}r)$$

For all $p \in R$, we have $p \leq r[n] \leq r$. Hence, $\operatorname{alph}(r[-n]) \subseteq \bigcap_{p\in R} \operatorname{alph}(p^{-1}r)$. Conversely, let $a \notin \operatorname{alph}(r[-n])$ and let q be the least prefix or r such that $|q|_a = |r|_a$. Since $\operatorname{alphinf}(r) \subseteq \operatorname{alph}(r[-n])$, we have $|q|_a = |r|_a < \infty$ and the trace q is either empty or prime. Moreover, $q \leq r[n] = \bigsqcup R$. Since $R = \downarrow R$ is downward closed, we deduce that $q \in R$. Therefore, $\bigcap_{p\in R} \operatorname{alph}(p^{-1}r) \subseteq \operatorname{alph}(q^{-1}r)$ and the claim follows from $a \notin \operatorname{alph}(q^{-1}r)$.

Using these n^{th} approximations, we give now the definition of the metric on α-traces.

Definition 7. *Let $x, y \in \mathbb{F}(\Sigma, D)$ be α-traces. The distance of x and y is $d(x, y) = 2^{-\ell(x,y)}$ where*

$$\ell(x, y) = \inf\{n \in \mathbb{N} \mid x[n] \neq y[n]\} \in \mathbb{N} \cup \{\infty\}$$

Let $x \in \mathbb{F}(\Sigma, D)$ and let $n \in \mathbb{N}$. The n^{th} open ball centered on x is

$$B(x, n) = \{y \in \mathbb{F}(\Sigma, D) \mid \ell(x, y) > n\}.$$

By convention, $\inf(\emptyset) = \infty$ and $2^{-\infty} = 0$. Hence, for all $x \in \mathbb{F}$, $d(x, x) = 0$. Conversely, let $x, y \in \mathbb{F}$ and assume that $d(x, y) = 0$. Then for all $n \in \mathbb{N}$, $x[n] = y[n]$ and we deduce that $x = \bigsqcup_{n \geq 0} x[n] = \bigsqcup_{n \geq 0} y[n] = y$. Clearly, the distance d is symmetric and for all $x, y, z \in \mathbb{F}$, it holds $d(x, z) \leq \max(d(x, y), d(y, z))$. Hence d is an ultra-metric distance on $\mathbb{F}$. Moreover, the family of open balls

$$\mathcal{B} = \{B(x, n) \mid x \in \mathbb{F} \text{ and } n \in \mathbb{N}\}$$

is an open basis for $(\mathbb{F}, d)$.

We will now show that the d-topology on $\mathbb{F}$ coincides with the $\sqsubseteq$-Lawson topology on $\mathbb{F}$. We first recall some definitions. Let $(T, \mathcal{O})$ be a topological space. A family $\mathcal{B} \subseteq \mathcal{O}$ is a basis of the topology if every open set can be written as some union of open sets from the basis $\mathcal{B}$. A family $\mathcal{S} \subseteq \mathcal{O}$ is a sub-basis of open sets if the family of finite intersections of sets from $\mathcal{S}$ is a basis of the topological space $(T, \mathcal{O})$.

Let $(Z, \leq)$ be a domain, i.e., an algebraic dcpo. The family

$$\mathcal{S} = \{\uparrow k \mid k \text{ is compact}\} \cup \{Z \setminus \uparrow k \mid k \text{ is compact}\}$$

is a sub-basis of the Lawson topology on $(Z, \leq)$. We have given the definition of the Lawson topology for domains only. For a more general definition, the reader is referred to [14].

Proposition 9. *The topology on $\mathbb{F}(\Sigma, D)$ induced by d is the Lawson topology on $\mathbb{F}(\Sigma, D)$ induced by the approximation order.*

Proof. Let $x = (r, A) \in \mathbb{F}_f$ be a compact α-trace and let $n = |x| = |r| < \infty$. Let $y, z \in \mathbb{F}$ be such that $\ell(y, z) > n$. Then,

$$y \in \uparrow x \iff x \sqsubseteq y[n] \iff x \sqsubseteq z[n] \iff z \in \uparrow x$$

Hence, for all $y \in \mathbb{F}$, either $B(y, n) \subseteq \uparrow x$ or $B(y, n) \subseteq \mathbb{F} \setminus \uparrow x$. Therefore, $\uparrow x$ and $\mathbb{F} \setminus \uparrow x$ are both d-open and d-closed. The d-topology on $\mathbb{F}$ is thus finer than the $\sqsubseteq$-Lawson topology on $\mathbb{F}$.

Claim. Let $x \in \mathbb{F}$ and let $n \in \mathbb{N}$. Then,

$$B(x, n) = \uparrow x[n] \setminus \uparrow\{z[n] \mid z[n] \not\sqsubseteq x[n]\}.$$

Let $y \in B(x, n)$. We have $x[n] = y[n] \sqsubseteq y$. Hence $B(x, n) \subseteq \uparrow x[n]$. Now assume that $y \in \uparrow z[n]$ for some $z \in \mathbb{F}$. From $z[n] \sqsubseteq y$, we deduce easily that $z[n] \sqsubseteq y[n] = x[n]$. Therefore, $B(x, n) \cap \uparrow\{z[n] \mid z[n] \not\sqsubseteq x[n]\} = \emptyset$.

Conversely, let $y \in \uparrow x[n] \setminus \uparrow\{z[n] \mid z[n] \not\sqsubseteq x[n]\}$. From $x[n] \sqsubseteq y$, we deduce that $x[n] \sqsubseteq y[n]$. Now, since $y \in \uparrow y[n]$ and $y \notin \uparrow\{z[n] \mid z[n] \not\sqsubseteq x[n]\}$ we deduce that $y[n] \sqsubseteq x[n]$. Therefore, $x[n] = y[n]$, i.e., $y \in B(x, n)$ and the claim is proved.

From the claim and the definition of the Lawson topology, we deduce that the d-open balls are also $\sqsubseteq$-Lawson open.

From Proposition 9, we will infer several properties which are general facts for the Lawson topology. First, let $(Z,\le)$ be a dcpo and let $(x_n)_{n\ge0}\subseteq Z$ be an increasing sequence. Then, the sequence $(x_n)_{n\ge0}$ converges to $\bigsqcup_{n\ge0}x_n$ for the Lawson topology see [14, 2].

Corollary 3. *Let $(x_n)_{n\ge0}\subseteq\mathbb{F}(\Sigma,D)$ be an $\sqsubseteq$-increasing sequence of α-traces. Then, the sequence $(x_n)_{n\ge0}$ converges in $(\mathbb{F}(\Sigma,D),d)$ and*

$$\lim_{n\to\infty}x_n=\bigsqcup_{n\ge0}x_n.$$

We give now the most important corollary of Proposition 9. Let $(Z,\le)$ be a poset which is both bounded complete and directed complete. Then, the Lawson topology is quasi-compact and T_1. If in addition, $(Z,\le)$ is algebraic then the Lawson topology is also T_2 (Hausdorff), whence compact, and the set $K(Z)$ of compact elements of Z is dense in Z. For all these properties, the reader is referred to [14] where they are stated and proved for slightly more general posets, namely continuous complete semi-lattices. The reader may also consult [2].

Theorem 5. *The metric space $(\mathbb{F}(\Sigma,D),d)$ is compact and complete. The set $\mathbb{F}_f(\Sigma,D)$ of finite α-traces is dense, discrete, and open in $(\mathbb{F}(\Sigma,D),d)$.*

Proof. By Theorems 2 and 3, the poset $(\mathbb{F},\sqsubseteq)$ is a bounded complete domain. Since the d-topology is the $\sqsubseteq$-Lawson topology, it follows that $(\mathbb{F},d)$ is a compact (whence complete) metric space and that the set $\mathbb{F}_f$ of compact α-traces (Theorem 3) is dense in $(\mathbb{F},d)$.

Hence, it remains to prove that $\mathbb{F}_f$ is discrete (this is not a general fact for the Lawson topology). Let $x=(r,A)\in\mathbb{F}_f$ be a finite α-trace and let $n=|x|=|r|$. We claim that $B(x,n+1)=\{x\}$. Indeed, let $y=(s,B)\in B(x,n+1)$. We deduce from Proposition 8 that $r[n]=s[n]$ and $r[n+1]=s[n+1]$. Since $|r|=n$, we have $r=r[n]=r[n+1]$. Hence, $s[n]=s[n+1]$ and $|y|=|s|\le n$. Therefore, $y=y[n]=x[n]=x$ which proves the claim.

We deduce directly from this claim that $\mathbb{F}_f$ is discrete, whence open.

We have seen in the previous section (Proposition 7) that the concatenation of α-traces is continuous with respect to the approximation order. We prove here a stronger result: the concatenation is uniformly continuous for the Lawson topology induced by the approximation order. This will be an immediate consequence of the following lemma.

Lemma 4. *For all α-traces $x,y\in\mathbb{F}$ and for all $n\ge0$, it holds*

$$(x\cdot y)[n]=(x[n]\cdot y[n])[n].$$

Proof. Let $x,y\in\mathbb{F}$ be α-traces. Since $x[n]\sqsubseteq x$ and $y[n]\sqsubseteq y$, we have by Thm. 1 that $x[n]\cdot y[n]\sqsubseteq x\cdot y$, and it follows $(x[n]\cdot y[n])[n]\sqsubseteq(x\cdot y)[n]$. Conversely, we claim that $z\sqsubseteq x[n]\cdot y[n]$ for all $z\sqsubseteq x\cdot y$ such that $|z|\le n$. Note that this claim allows to conclude the proof of the lemma since clearly it implies $(x\cdot y)[n]\sqsubseteq(x[n]\cdot y[n])[n]$.

In order to prove this claim, assume that $x=(r,A)$, $y=(s,B)$ and let $z=(t,C)$ be a finite α-trace of length at most n such that $z\sqsubseteq x\cdot y$. By definition of the approximation order and of the concatenation on α-traces, we have $t \leq r\mu_A(s)$ and $A \cup B \cup \sigma_A(s) \cup \mathrm{alph}(t^{-1}r\mu_A(s)) \subseteq C$. Using the generalization of Levi's lemma to real traces (Sect. 2.2) we obtain $t = r's'$ for some $r', s' \in \mathbb{M}$ such that $r' \leq r$, $s' \leq \mu_A(s)$ and $\mathrm{alph}(s') \times \mathrm{alph}(r'^{-1}r) \subseteq I$.

To simplify the notations, we let $A' = A \cup \mathrm{alph}(r[-n])$ and $B' = B \cup \mathrm{alph}(s[-n])$ so that $x[n] = (r[n], A')$ and $y[n] = (s[n], B')$. Note that, since $|t| = |r's'| \leq n$, we have $r' \leq r[n] \leq r$ and $A' \subseteq A \cup \mathrm{alph}(r'^{-1}r)$. Similarly, $s' \leq s[n] \leq s$ and $B' \subseteq B \cup \mathrm{alph}(s'^{-1}s)$.

We have to prove that $z \sqsubseteq x[n] \cdot y[n]$, i.e.,

$$t \leq r[n]\mu_{A'}(s[n]) \tag{5}$$

and

$$A' \cup B' \cup \sigma_{A'}(s[n]) \cup \mathrm{alph}(t^{-1}r[n]\mu_{A'}(s[n])) \subseteq C. \tag{6}$$

To prove (5) we use again the generalization of Levi's lemma to real traces (Sect. 2.2). We thus have to prove that

$$r' \leq r[n], s' \leq \mu_{A'}(s[n]) \text{ and } \mathrm{alph}(s') \times \mathrm{alph}(r'^{-1}r[n]) \subseteq I.$$

We have already seen that $r' \leq r[n] \leq r$ and $s' \leq s[n] \leq s$. From $A' \subseteq A \cup \mathrm{alph}(r'^{-1}r)$ and $\mathrm{alph}(s') \times (A \cup \mathrm{alph}(r'^{-1}r)) \subseteq I$, we deduce that $\mathrm{alph}(s') \times \mathrm{alph}(r'^{-1}r[n]) \subseteq I$ and $s' \leq \mu_{A'}(s[n])$, which completes the proof of (5).

Now, we have $r' \leq r[n] \leq r$ and $s' \leq \mu_{A'}(s[n]) \leq s[n] \leq s$. Using $t^{-1}r\mu_A(s) = (r'^{-1}r)(s'^{-1}\mu_A(s))$ and $t^{-1}r[n]\mu_{A'}(s[n]) = (r'^{-1}r[n])(s'^{-1}\mu_{A'}(s[n]))$ we deduce that

$$\begin{aligned}
A' \cup B' \cup \sigma_{A'}(s[n]) \cup \mathrm{alph}(t^{-1}r[n]\mu_{A'}(s[n])) \\
\subseteq \quad & A \cup \mathrm{alph}(r'^{-1}r) \cup B \cup \mathrm{alph}(s'^{-1}s) \\
= \quad & A \cup B \cup \sigma_A(s) \cup \mathrm{alph}(t^{-1}r\mu_A(s)) \\
\subseteq \quad & C,
\end{aligned}$$

which proves (6).

Proposition 10. *The concatenation on α-traces is uniformly continuous. More precisely, let $x, x', y, y' \in \mathbb{F}(\Sigma, D)$ be α-traces. Then*

$$d(xy, x'y') \leq \max(d(x, x'), d(y, y')).$$

Proof. From Lemma 4, we deduce that the set of approximations of length at most n of xy depends only on the n^{th} approximations of x and y. Therefore, for all $n \leq 0$, $x[n] = x'[n]$ and $y[n] = y'[n]$ imply $(xy)[n] = (x'y')[n]$ which proves the proposition.

Note that Prop. 7 is in fact a corollary of Prop. 10. Indeed, a Lawson- continuous monotone mapping between two domains is necessarily $\bigsqcup$-continuous with respect to the partial orders of the posets. However, the converse is false in general. Hence, we cannot infer the uniform continuity of the concatenation from its $\bigsqcup$-continuity and the compactness of $(\mathbb{F}, d)$.

5.2. Projective limits

Most of the results of this section can also be obtained by means of projective limits. Let us consider the sets $\mathbb{F}[n] = \{x[n] \mid x \in \mathbb{F}\}$. Thanks to Lem. 4, we see that the operation $\cdot_n$ defined on $\mathbb{F}[n]$ by $x[n] \cdot_n y[n] = (x \cdot y)[n]$ is well-defined. It is clearly associative with $(1, \emptyset)$ as neutral, whence $\mathbb{F}[n]$ is a monoid. Now, for $n \geq m$, let $\varphi_{n,m} : \mathbb{F}[n] \longrightarrow \mathbb{F}[m]$ be defined by $\varphi_{n,m}(x[n]) = x[m]$. This mapping is well-defined since $x[m] = x[n][m]$ for $n \geq m$. Moreover, for $n \geq m \geq p$ we clearly have $\varphi_{n,p} = \varphi_{m,p} \circ \varphi_{n,m}$ hence these mappings define a projective system. Finally, it is easy to see that the mappings $\varphi_{n,m}$ for $n \geq m$ are morphisms. Therefore, the projective limit

$$\varprojlim \mathbb{F}[n] = \{(x_n)_{n\geq 0} \in \prod_{n\geq 0} \mathbb{F}[n] \mid \varphi_{n,m}(x_n) = x_m \text{ for all } n \geq m\}$$

is a monoid with the operation $(x_n)_{n\geq 0} \cdot (y_n)_{n\geq 0} = (x_n \cdot_n y_n)_{n\geq 0}$. We claim that this projective monoid is actually isomorphic to the monoid of α-traces. Indeed, we have clearly a morphism $\varphi : \mathbb{F} \longrightarrow \varprojlim \mathbb{F}[n]$ defined by $\varphi(x) = (x[n])_{n\geq 0}$ for $x \in \mathbb{F}$. Since $x = \bigsqcup_{n\geq 0} x[n]$, the morphism φ is injective. To show that it is also onto, let $(x_n)_{n\geq 0} \in \varprojlim \mathbb{F}[n]$. Since this is a projective sequence, we have $x_m = \varphi_{n,m}(x_n) = x_n[m] \sqsubseteq x_n$ for all $n \geq m$. Hence, the sequence $(x_n)_{n\geq 0}$ is increasing and we can consider $x = \bigsqcup_{n\geq 0} x_n \in \mathbb{F}$. We claim that $\varphi(x) = (x_n)_{n\geq 0}$ which shows that φ is surjective. Indeed, let $m \geq 0$, the α-trace $x[m]$ is compact and $x[m] \sqsubseteq x = \bigsqcup_{n\geq 0} x_n$. Hence, $x[m] \sqsubseteq x_n$ for some $n \geq m$ and we obtain $x[m] \sqsubseteq x_n[m] = x_m$. Conversely, $x_m \sqsubseteq x$ and we get $x_m = x_m[m] \sqsubseteq x[m]$. Therefore, $x[m] = x_m$ which proves the claim.

Now, the sets $\mathbb{F}[n]$ for $n \geq 0$ are finite, whence compact for the discrete topology and the morphisms $\varphi_{n,m}$ for $n \geq m$ are continuous. Therefore, the projective limit $\varprojlim \mathbb{F}[n]$ is compact for the product topology. Moreover, the concatenations $\cdot_n$ for $n \geq 0$ are also continuous, and it follows that the concatenation on $\varprojlim \mathbb{F}[n]$ is continuous, whence uniformly continuous since $\varprojlim \mathbb{F}[n]$ is compact.

Now, the discrete topology on $\mathbb{F}[n]$ is induced by the ultra-metric d_n defined by

$$d_n(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Hence, the product topology on $\varprojlim \mathbb{F}[n]$ is induced by the ultra-metric distance δ defined for $x, y \in \mathbb{F}$ by

$$\delta(x, y) = \frac{1}{2} \sum_{n\geq 0} 2^{-n} d_n(x[n], y[n]) = \frac{1}{2} \sum_{n\geq \ell(x,y)} 2^{-n} = 2^{-\ell(x,y)}.$$

Therefore, identifying $\mathbb{F}$ with $\varprojlim \mathbb{F}[n]$ via φ we can see that the product topology on $\varprojlim \mathbb{F}[n]$ is precisely the topology of $\mathbb{F}$ induced by the metric d of Def 7.

6. Delta-Traces

One might object that the domain of α-traces is not abstract enough. If we consider the full dependence relation $D = \Sigma \times \Sigma$ then we are in the sequential case and the real part of an α-trace is simply a word: $\mathbb{F}(\Sigma, \Sigma\times\Sigma) \subseteq \Sigma^\infty \times \mathcal{P}(\Sigma)$. Indeed, it has all the good properties seen in the previous sections but this is also the case of the more abstract sequential model $\Sigma^*\bot \cup \Sigma^\infty$ discussed at the beginning of Sect. 3: $(\Sigma^*\bot \cup \Sigma^\infty, \sqsubseteq)$ is a prime algebraic coherently complete poset; the compacts, which coincide with the primes, are the finite behaviours; the concatenation is monotone and continuous with respect to the approximation ordering.

Intuitively, in the sequential case, when a behaviour is not terminated it is not necessary to know which actions may still occur because, whatever these possible actions are, we cannot start the execution of another process. In other words, in $\mathbb{F}(\Sigma, \Sigma\times\Sigma)$ the relation $(r, A) \equiv (s, B)$ if $r = s$ and either $A = B = \emptyset$ or $A \neq \emptyset \neq B$ is a congruence with respect to the concatenation.

This remark can be extended to the non-sequential case. If a behaviour x is not terminated, one can immediately start all actions independent from the actions which may still occur in x. Therefore, if a process p has already performed the real behaviour r and A contains all actions which may still occur in p, then it is sufficient to know $D(A) = \{b \in \Sigma \mid (b, a) \in D \text{ for some } a \in A\}$ to decide which actions can be started concurrently to the end of the process p. In other words, the relation on $\mathbb{F}(\Sigma, D)$ defined by $(r, A) \equiv (s, B)$ if $r = s$ and $D(A) = D(B)$ is a congruence for the concatenation. This yields to the following definition.

Definition 8. *Let (Σ, D) be a dependence alphabet. A δ-trace over (Σ, D) is an ordered pair $(r, D(A))$ where $r \in \mathbb{R}(\Sigma, D)$ is a real trace and $A \subseteq \Sigma$ is such that $D(\mathrm{alphinf}(r)) \subseteq D(A)$. The set of δ-traces is denoted by $\mathbb{F}_\delta(\Sigma, D)$ or simply by $\mathbb{F}_\delta$. The concatenation on $\mathbb{F}_\delta(\Sigma, D)$ is defined by*

$$(r, D(A)) \cdot (s, D(B)) = (r\mu_A(s), D(A \cup B \cup \sigma_A(s)))$$

and the approximation relation is defined by

$$(r, D(A)) \sqsubseteq (s, D(B)) \text{ if } r \leq s \text{ and } D(B \cup \mathrm{alph}(r^{-1}s)) \subseteq D(A)$$

Note that the concatenation is well-defined since for all real traces s, both $\mu_A(s)$ and $\sigma_A(s)$ depend on $D(A)$ only. $(\mathbb{F}_\delta(\Sigma, D), \cdot)$ is a monoid with $(1, \emptyset)$ as neutral and $(\mathbb{F}_\delta(\Sigma, D), \sqsubseteq)$ is a poset with $(1, \Sigma)$ as bottom element. Moreover, the concatenation of δ-traces is monotone for the approximation order.

If $D = \Sigma \times \Sigma$ is the full dependence relation then the consideration of $\mathbb{F}_\delta(\Sigma, \Sigma\times\Sigma)$ is equivalent to the formalism of sequential words together with the non-termination symbol $\bot$. In fact, for $D(A)$ we only have two possibilities: either $D(A) = \Sigma$, which means non-termination, or $D(A) = \emptyset$, which means explicit termination. A δ-trace in this special case either is a finite non-terminated word $v\bot$, with $v \in \Sigma^*$, or it is a finite terminated word $v \in \Sigma^*$, or it is an infinite word $v \in \Sigma^\omega$ (where the non-termination symbol is redundant and may be omitted). Therefore $\mathbb{F}_\delta(\Sigma, \Sigma \times \Sigma) = \Sigma^*\bot \cup \Sigma^\infty$. Moreover, the calculus of δ-traces is as usual such that $v\bot$ or infinite words are right-absorbent. Furthermore, for all $u, v \in \Sigma^\infty$, we have $u\bot \sqsubseteq v\bot \sqsubseteq v$ if and only if $u \leq v$.

Even if $\mathbb{F}_\delta(\Sigma, D)$ is more abstract than $\mathbb{F}(\Sigma, D)$, we have concentrated our attention on the latter because we believe that it is more intuitive and easier to deal with. Moreover, most properties (but not all) can be transfered from $\mathbb{F}(\Sigma, D)$ to $\mathbb{F}_\delta(\Sigma, D)$ using the canonical surjective and monotone morphism $\phi : \mathbb{F}(\Sigma, D) \longrightarrow \mathbb{F}_\delta(\Sigma, D)$ defined by $\phi(r, A) = (r, D(A))$. Finally, as we will see below, the domain $(\mathbb{F}_\delta(\Sigma, D), \sqsubseteq)$ is less satisfactory than $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ since it is neither distributive nor prime algebraic in general.

Let us sketch the main steps which allow to transfer results from α-traces to δ-traces. We mainly use general facts from poset theory which are easy to prove. If needed, the reader should be able to reconstruct the detailed proofs from the skeleton below.

Note that, we can also embed $(\mathbb{F}_\delta(\Sigma, D), \sqsubseteq)$ into $(\mathbb{F}(\Sigma, D), \sqsubseteq)$ with the monotone mapping ψ defined by $\psi(r, D(A)) = (r, A_{\max})$ where $A_{\max} = \{a \in \Sigma \mid D(a) \subseteq D(A)\}$. We have $\phi \circ \psi = \mathrm{Id}$ but ψ is not a morphism. For instance, if $(\Sigma, D) = a$ —— b —— c then $\psi(1, D(a)) \cdot \psi(1, D(c)) = (1, \{a, c\}) \neq (1, \Sigma) = \psi((1, D(a)) \cdot (1, D(c)))$.

Let $Y \subseteq \mathbb{F}_\delta(\Sigma, D)$ be a coherent set of δ-traces. Then, $X = \psi(Y) \subseteq \mathbb{F}(\Sigma, D)$ is coherent and we can verify that $\sqcup Y$ exists and

$$\sqcup Y = \phi(\sqcup X) = (\sqcup \mathrm{Re}(Y), D(\{a \in \Sigma \mid D(a) \subseteq \mathrm{Im}(y) \text{ for all } y \in Y\})).$$

Therefore, $(\mathbb{F}_\delta(\Sigma, D), \sqsubseteq)$ is a coherently complete poset.

The mappings ϕ and ψ are both $\sqcup$-continuous. This allows to deduce that a δ-trace y is compact if and only if $\psi(y)$ is a compact α-trace, i.e., if and only if y is finite. Then, we easily obtain that $(\mathbb{F}_\delta(\Sigma, D), \sqsubseteq)$ is algebraic, whence a coherently complete domain.

The situation is not as good for the prime algebraicity. First, the mapping ψ is $\sqcup$-continuous even for coherent sets: We have $\psi(\sqcup X) = \sqcup \psi(X)$ for all coherent sets $X \subseteq \mathbb{F}_\delta(\Sigma, D)$. This is not the case for the mapping ϕ (consider $\Sigma = \{a, b, c\}$, $D = \Sigma \times \Sigma$ and $X = \{(1, a), (1, b)\}$). Therefore if $y \in \mathbb{F}_\delta(\Sigma, D)$ is such that $\psi(y)$ is a prime α-trace, then we can deduce that y is a prime δ-trace. But we cannot infer the converse: let $(\Sigma, D) = a$ —— b —— c, then $y = (1, D(a)) \in \mathbb{F}_\delta(\Sigma, D)$ is a prime δ-trace while $\psi(y) = (1, a)$ is not a prime α-trace. Moreover, the domain of δ-trace is not prime algebraic in general. We have the following counterexample: Let $(\Sigma, D) = a$ —— b —— c —— d be the line of four letters. Then the finite δ-trace $(1, D(a))$ is not the least upper bound of its prime approximations. The approximations of $(1, D(a))$ are in fact $(1, \Sigma) \sqsubset (1, D(a, b)) \sqsubset (1, D(a))$, and its only prime approximation is $(1, D(a, b))$. Note that $(1, D(a))$ is not prime since $(1, D(a)) \sqsubseteq (1, D(b)) \vee (1, D(c)) = (1, \emptyset)$ but $(1, D(a)) \not\sqsubseteq (1, D(b))$ and $(1, D(a)) \not\sqsubseteq (1, D(c))$.

Finally, since they do not use the fact that $\mathbb{F}(\Sigma, D)$ is prime algebraic, all results of Sect. 5 extend easily to δ-traces. The n^{th} approximation of a δ-trace $x = (r, D(A))$ is

$$\begin{aligned} x[n] &= \sqcup\{y \in \mathbb{F}_\delta(\Sigma, D) \mid y \sqsubseteq x \text{ and } |y| \leq n\} \\ &= (r[n], A \cup D(\{a \in \Sigma \mid D(a) \subseteq D(p^{-1}r) \text{ for all } p \leq r, |p| \leq n\})) \end{aligned}$$

The ultra-metric distance $d_\delta(x, y) = 2^{-\ell_\delta(x,y)}$ where

$$\ell_\delta(x, y) = \inf\{n \in \mathbb{N} \mid x[n] \neq y[n]\} \in \mathbb{N} \cup \{\infty\}$$

induces the Lawson topology on $\mathbb{F}_\delta(\Sigma, D)$. Hence, $(\mathbb{F}_\delta(\Sigma, D), d_\delta)$ is a compact and complete metric space. The set of finite δ-traces is dense, discrete and open in $\mathbb{F}_\delta(\Sigma, D)$. Finally, the concatenation of δ-traces is uniformly continuous for d_δ, whence it is also $\sqcup$-continuous. The proofs of all these results are quite similar to the proofs of the corresponding results in Sect. 5.

7. Conclusion

We have seen that α-traces have many properties being useful for a modular semantics of concurrent systems and classical concurrent languages like CSP or CCS [16, 22].

On the other hand, α-traces are a more concrete than necessary, while δ-traces are a slightly too abstract. The characterization of the dependence alphabets where $\mathbb{F}_\delta(\Sigma, D)$ is prime algebraic is still an open question. Moreover, it would be interesting to give a characterization of prime δ-traces.

Finally, it might be possible to find some other suitable domain intermediary between α-traces and δ-traces.

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