SOME IDENTITIES RELATED TO AUTOMATA, DETERMINANTS, AND MÖBIUS FUNCTIONS

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We define the Möbius function for a language S being closed under factors as the formal inverse of the characteristic series over the set S. We derive identities in commuting and non-commutating variables characterizing this function as a quotient of polynomials which can be expressed as certain determinants. These determinants in turn are obtained by some matrix related to the minimal automaton recognizing S. Our contribution extends some recent work of Choffrut and Goldwurm.

1 Introduction

Möbius functions play an important role in combinatorics. For monoids they were introduced by Cartier and Foata [3] leading to a generalization of MacMahon's Master Theorem. The spirit of this theorem is an expression of certain series as the formal inverse of some determinant. The work of Cartier and Foata led also to a systematic study of free partially commutative monoids. These monoids are of particular interest in computer science due to the possibility to model basic phenomena on concurrent processes [10, 7].
In 1996 Choffrut and Goldwurm [4] proved that (an unambiguous lifting of) the Möbius function of a free partially commutative monoid can be expressed as the determinant of the minimal automaton recognizing the set of its lexicographic normal forms if and only if the independence relation has a transitive orientation. Taking a slightly different viewpoint this result is in fact a statement about languages defined by forbidden factors of length exactly two. The aim of this paper is a generalization to other classes of languages defined by forbidden factors.

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We define the Möbius function μ_S for languages $S = X^* \setminus X^*FX^*$ being closed under factors as the formal inverse of the characteristic series over S. The dream would be a general formula of the type

$$\mu_S = \mathrm{Det}(\mathbf{1} - M)$$

where M is a matrix related to the minimal automaton recognizing S. Such an identity fails in general, but we obtain a closely related identity. Informally, our main results can be stated as follows: (The precise meaning of the notions involved will be given below.)

Theorem *Let $F \subseteq X^*$ be a finite set of forbidden factors, $1 \notin F$, and let $l = \max\{|u| \mid u \in F\}$, $l \geq 2$, the maximal length. Let $S = X^* \setminus X^*FX^*$ and M be the $n \times n$-matrix associated to the minimal finite automaton recognizing the language S (c.f. Sect. 4). Then we find explicitly a polynomial $D(S, l-1)$ of degree at most $n \cdot (l-1) \cdot \max\{1, l-2\}$ such that we have the following identity in commuting variables:*

$$S \cdot \mathrm{Det}(\mathbf{1} - M) \equiv_c D(S, l-1).$$

Moreover, we derive a graph theoretical criterion implying the following identity of formal power series:

$$S \cdot \mathrm{Det}(\mathbf{1} - M) = D(S, l-1).$$

Thus, the product

$$\mathrm{Det}(\mathbf{1} - M) \cdot D(S, l-1)^{-1}$$

is an explicit description of the Möbius function of S, if the criterion is satisfied. It turns out that, if all forbidden factors are of the form ba for some letters a, b with $a < b$ (for some order $<$ of the alphabet), then the polynomial $D(S, l-1)$ equals 1 and the criterion is satisfied. Since it is well-known that the set of lexicographic normal forms of a free partially commutative monoid has a description by forbidden factors of length two if and only if the independence relation is equipped with a transitive orientation, we obtain the setting of Choffrut and Goldwurm as a special case.
The outline of the paper is as follows. In Sect. 2 we recall some well-known facts on formal power series, Möbius functions, and overlapping chains. Since our calculations involve determinants over a non-commutative ring (being perhaps not standard) we develop the necessary background in Sect. 3. The new results start with Sect. 4. In the final section we illustrate our method again by an example for some partial commutation over four letters.

2 Formal power series, Möbius functions, and overlapping chains

Let X be a finite alphabet. The *ring of formal power series* $\mathbf{Z}\langle\langle X \rangle\rangle$ is the set of mappings $f : X^* \longrightarrow \mathbf{Z}$ where addition and multiplication are defined by:

$$(f+g)(w) = f(w) + g(w),$$

$$(f \cdot g)(w) = \sum_{uv=w} f(w) \cdot g(w).$$

It is easy to see that a power series $f \in \mathbf{Z}\langle\langle X^* \rangle\rangle$ has a formal inverse f^{-1} if and only if $f(1) \in \{-1, +1\}$. A *polynomial* is a formal power series f with a finite support, i.e., the set $\{w \in X^* \mid f(w) \neq 0\}$ is finite. The *degree* of a non-zero polynomial f is the maximal length of a word in the support, $\max\{|w| \mid f(w) \neq 0\}$.

By 1 we denote the empty word $1 \in X^*$ and the unit element of $\mathbf{Z}\langle\langle X \rangle\rangle$ (or any other ring). For $\mathbf{Z}\langle\langle X \rangle\rangle$ the unit element is the characteristic series over $\{1\}$. The symbol $\mathbf{1}$ (written in boldface) is reserved to denote the $(n \times n$-)unit-matrix over some ring.

A subset $S \subseteq X^*$ is frequently identified with its characteristic series, $S = \zeta_S$, where:

$$\zeta_S(w) = 1 \text{ if } w \in S \text{ and } \zeta_S(w) = 0 \text{ otherwise.}$$

By abuse of language, we say that two power series $g, h \in \mathbf{Z}\langle\langle X \rangle\rangle$ are *equal in commuting variables*, written as $g \equiv_c h$, if $g - h$ is an element in the two-sided ideal generated by the series $f \cdot a - a \cdot f$ where $a \in X$ and $f \in \mathbf{Z}\langle\langle X \rangle\rangle$. In particular, we have

$$f \cdot p \equiv_c p \cdot f \text{ for all } f, p \in \mathbf{Z}\langle\langle X \rangle\rangle \text{ where } p \text{ is a polynomial.}$$

Definition 1 *Let S be a non-empty language $S \subseteq X^*$ being closed under factors, i.e., $uvw \in S$ implies $v \in S$. The Möbius function μ_S is defined to be the formal inverse of the characteristic series over S. Hence:*

$$\mu_S = (\sum_{w \in S} w)^{-1} = S^{-1}.$$

Remark 1 *Let $\mathbf{M}$ be a quotient monoid of X^* where the Möbius function $\mu_{\mathbf{M}}$ is defined in the sense of Cartier and Foata, see [3], and let $S \subseteq X^*$ be a cross-section of $\mathbf{M}$ being closed under factors. (Such a set $S \subseteq X^*$ always exists, e. g. take S to be the set of lexicographical normal forms.) Then $\mu_S \in \mathbf{Z}\langle\langle X \rangle\rangle$ is a lifting (or a representation) of the Möbius function $\mu_{\mathbf{M}}$.*

Any non-empty language $S \subseteq X^*$ being closed under factors can be defined by some *set of forbidden factors* $F \subseteq X^*$ such that $S = X^* \setminus X^*FX^*$. Since $S \neq \emptyset$, we have $1 \notin F$ and we may assume that F is a basis of the ideal X^*FX^*, i. e., $X^*FX^* = X^*F'X^*$ implies $F \subseteq F'$.

Definition 2 *Let* $1 \in S = X^* \setminus X^*FX^*$ *and* F *be a basis of* X^*FX^*. *An* overlapping chain *is a sequence* $(w_1, \ldots, w_n)$ *such that* $n \geq 0$, $w_1 \in S \cap X$, $w_i \in S$, *and* $w_{i-1}w_i \in X^*F \setminus X^*FX^+$ *for* $1 < i \leq n$.

The concept of overlapping chains appears in [1], see also [8, 9]. Independently it appears also in [6], where Prop. 2 is proved. A closely related result on Hilbert series can also be found in [2, Thm. 5.1], in that paper overlapping chains are called *worms*.

Proposition 2 ([6], Thm 4.4.2) *Let* C *be the set of overlapping chains. Then we have*

$$\mu_s = \sum_{(w_1,\ldots,w_n)\in\mathcal{C}} (-1)^n w_1 \cdots w_n.$$

Corollary 3 *If the Möbius function* μ_S *is a polynomial, then* S *is defined by some finite set* F *of forbidden factors. If* F *is finite, then* μ_S *is a polynomial if and only if there is a maximal integer* n *where some* $(w_1, \ldots, w_n) \in \mathcal{C}$ *exists.*

Proof of Cor. 3: By [6, Lem. 4.4.1] two overlapping chains $(w_1, \ldots, w_n) \in \mathcal{C}$ and $(w'_1, \ldots, w'_n) \in \mathcal{C}$ are equal if and only if $w_1 \cdots w_n = w'_1 \cdots w'_n$ denote the same words. Hence, by Prop. 2 the power series μ_S is a polynomial if and only if $\mathcal{C}$ is a finite set. The result follows since $F \setminus X$ is a subset of the support of μ_S and X is finite.

Example 1

i) Let $F = \{abb\}$, *then we have*

$$\mu_S = 1 - X + abb.$$

ii) Let $F = \{aba\}$, *then we have*

$$\mathcal{C} = \{(1), (x), (a, ba), (a, ba, ba), \ldots, (a, ba, ba, \ldots, ba), \ldots \mid x \in X\}.$$

Hence

$$\begin{aligned}
\mu_S &= 1 - X - \sum_{n\geq 1} (-1)^n a(ba)^n \\
&= 1 - X \setminus \{a\} - a(1 + ba)^{-1} \\
&= (1 - X + (1 - X \setminus \{a\})ba)(1 + ba)^{-1}.
\end{aligned}$$

Example 2 *Let $I \subseteq X \times X$ be a symmetric (independence) relation and let $\equiv_I$ be the congruence generated by $ab \equiv_I ba$ for $(a,b) \in I$. Let $<$ be a total order of X and $I^+ = (I \cap >)$. For $w \in X^*$ let* $\mathrm{lex}([w])$ *denote the lexicographically first element in the congruence class of w and let* Lex *be the set of lexicographical normal forms:* $\mathrm{Lex} = \{\mathrm{lex}([w]) \mid w \in X^*\}$.

i) Let I^+ be transitive. Then $\mathrm{Lex} = X^* \setminus X^*FX^*$ *where F is the finite set of forbidden factors $F = \{ba \mid (b,a) \in I^+\}$. In this case:*

$$\mu_{\mathrm{Lex}} = \sum_{(a_1,\ldots,a_n)\in\mathcal{C}} (-1)^n a_1 \cdots a_n.$$

Clearly, $(a_1,\ldots,a_n) \in \mathcal{C}$ if and only if $a_1 > \cdots > a_n$ and $(a_i,a_j) \in I$ for all $i \neq j$, $1 \leq i,j \leq n$ (since I^+ is transitive).

In the terminology of [5] the function μ_{Lex} is therefore an unambiguous lifting of the Möbius function of $X^*/\equiv_I$.

ii) If I^+ is not transitive, then there are letters $a,b,c \in X$ such that $(c,b) \in I^+$, $(b,a) \in I^+$, but $(c,a) \notin I^+$. Hence $ca^nb \notin$ Lex, *since* $\mathrm{lex}([ca^nb]) = bca^n$. *Every proper factor of the word ca^nb belongs to* Lex. *Therefore we have $\{ca^nb \mid n \geq 0\} \subseteq F$ and F must be infinite. As a consequence, μ_{Lex} is no polynomial by Cor. 3.*

3 Determinants over non-commutative rings

Let R be a ring with 1 being not necessarily commutative. There is no unique definition for the determinant of an $n \times n$ matrix over a non-commutative ring. For our calculations the following definition is convenient:

Definition 3 *Let $A = (a_{ij})$ be an $n \times n$ matrix with coefficients in R, $a_{ij} \in R$ for $1 \leq i,j \leq n$. Define the determinant of A by:*

$$\mathrm{Det}(A) = \sum_{\sigma} (-1)^{\sigma} a_{\sigma(1),1} \cdots a_{\sigma(n),n},$$

where the sum is taken over all permutations of $\{1,\ldots,n\}$ and $(-1)^\sigma$ denotes the sign of the permutation σ.
The trace of the matrix A is defined as usual by the sum over the diagonal elements:

$$\mathrm{Trace}(A) = \sum_{i=1}^{n} a_{i,i}.$$

Remark 4 *Taking the determinant of the transposed matrix we find the definition of a determinant as e.g. in [3, page 54].*

Calculations with determinants over a non-commutative ring need some care. For example consider a matrix where two columns are equal. We have:

$$\mathrm{Det}\begin{pmatrix} a & a \\ b & b \end{pmatrix} = ab - ba$$

which is not zero, unless $ab = ba$.
Moreover, determinants are not multiplicative, in general:

$$\mathrm{Det}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\right) = acbd,$$
$$\mathrm{Det}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\mathrm{Det}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = abcd.$$

Nevertheless there are some general rules being used in the following.

Rule 1 *The determinant is zero, if a row or a column is zero or if two rows are equal.*

Proof: Clearly, if a row or a column is zero, then the determinant is zero. Let A be a matrix where the i-th and j-th row are equal, $i \neq j$. Let τ be the transposition with $\tau(i) = j$. Then $\sigma \longmapsto \tau \circ \sigma$ defines an involution without fixed points and we have

$$(-1)^{\sigma} a_{\sigma(1),1} \cdots a_{\sigma(n),n} = -(-1)^{\tau\circ\sigma} a_{\tau\circ\sigma(1),1} \cdots a_{\tau\circ\sigma(n),n}$$

for all permutations σ. The result follows.

Rule 2 *The determinant does not change its value if we add or subtract some row from another one.*

Proof: The classical proof applies due to Rule 1.

For the rest of this section we fix the following notations:
A denotes any $n \times n$ matrix over R, $n \geq 2$.
B and C denote matrices where all rows are equal: $B = (b_{ij})$ and $b_{ij} = b_j$ for some vector $(b_1, \ldots, b_n) \in R^n$, $C = (c_{ij})$ and $c_{ij} = c_j$ for some vector $(c_1, \ldots, c_n) \in R^n$.

T denotes a matrix which is strictly upper triangular, $T = (t_{ij})$ and $i \geq j$ implies $t_{ij} = 0$.
D denotes a diagonal matrix where the diagonal elements are central in R. This means $D = (d_{ij})$, $d_{ij} = 0$ for $i \neq j$, $d_{ii} = d_i$ for some vector $(d_1, \ldots, d_n) \in R^n$ such that $d_i r = r d_i$ for all $r \in R$, $i \in \{1, \ldots, n\}$.
Finally, $\mathbf{1}$ denotes the $n \times n$-unit-matrix, i. e., $\mathbf{1}$ is a diagonal matrix as above where $d_i = 1$ for $1 \leq i \leq n$.
For a matrix A and $J \subseteq \{1, \ldots, n\}$ let A^J be the matrix obtained from A where all rows and columns from the index set J are canceled. We adopt the convention that $\mathrm{Det}(A^J) = 1$ for $J = \{1, \ldots, n\}$.

Rule 3 *We have*

$$\mathrm{Det}(D + A) = \sum_{J \subseteq \{1, \ldots, n\}} \left(\prod_{j \in J} d_j \right) \mathrm{Det}\left(A^J\right).$$

Proof: The classical proof applies since all d_i are central, $1 \leq i \leq n$.

Combining Rules 1 and 3 together with the observation that $\mathrm{Det}(B^J) = 0$ for $|J| \leq n - 2$, we obtain:

Rule 4 *We have*

$$\mathrm{Det}(\mathbf{1} + B) = 1 + \mathrm{Trace}(B) = 1 + \sum_{j=1}^{n} b_j.$$

Rule 5 *We have*

$$\mathrm{Det}(B - T) = b_1 t_{1,2} \cdots t_{n-1,n}.$$

Proof: This follows by induction on n since by Rule 2 we have:

$$\begin{aligned}
\mathrm{Det}(B - T) &= \mathrm{Det} \begin{pmatrix} b_1 & \cdots & b_{n-1} - t_{1,n-1} & b_n - t_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ b_1 & \cdots & b_{n-1} & b_n - t_{n-1,n} \\ b_1 & \cdots & b_{n-1} & b_n \end{pmatrix} \\
&= \mathrm{Det} \begin{pmatrix} b_1 & \cdots & b_{n-1} - t_{1,n-1} & b_n - t_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ b_1 & \cdots & b_{n-1} & b_n - t_{n-1,n} \\ 0 & \cdots & 0 & t_{n-1,n} \end{pmatrix}
\end{aligned}$$

$$= \quad \mathrm{Det}\begin{pmatrix} b_1 & \cdots & b_{n-1} - t_{1,n-1} \\ \vdots & \ddots & \vdots \\ b_1 & \cdots & b_{n-1} \end{pmatrix} \cdot t_{n-1,n}.$$

By Rules 3 and 5 we obtain:

Rule 6 *We have*

$$\mathrm{Det}(D + B - T) = \sum_{\substack{i_1 < \cdots < i_k \\ J = \{1,\ldots,n\} \setminus \{i_1,\ldots,i_k\}}} \left(\prod_{j \in J} d_j \right) b_{i_1} t_{i_1,i_2} \cdots t_{i_{k-1},i_k}.$$

Rule 7 *We have*

$$\mathrm{Det}(\mathbf{1} + B - T) = \sum_{i_1 < \cdots < i_k} b_{i_1} t_{i_1,i_2} \cdots t_{i_{k-1},i_k}.$$

Recall that determinants are not multiplicative, in general. Thus, for the following rule we need a proof.

Rule 8 *We have*

$$\mathrm{Det}((\mathbf{1} + B)(\mathbf{1} + C - T)) = \mathrm{Det}(\mathbf{1} + B)\,\mathrm{Det}(\mathbf{1} + C - T).$$

Proof: Clearly:

$$(\mathbf{1} + B)(\mathbf{1} + C - T) = \mathbf{1} + (B + BC - BT + C) - T$$

and $B + BC - BT + C$ is a matrix where all rows are equal. More precisely

$$(B + BC - BT + C)_{ij} = b_j + \sum_l b_l c_j - \sum_{l<j} b_l t_{lj} + c_j =: e_j.$$

By Rule 7 we obtain

$$\mathrm{Det}((\mathbf{1} + B)(\mathbf{1} + C - T)) = 1 + \sum_{\substack{i_1 < \ldots < i_k \\ k \geq 1}} e_{i_1} t_{i_1,i_2} \cdots t_{i_{k-1},i_k}.$$

On the other hand we have:

$$\mathrm{Det}(\mathbf{1} + B) = 1 + \sum_l b_l,$$

$$\mathrm{Det}(\mathbf{1}+C-T) = 1 + \sum_{\substack{i_1<\ldots<i_k \\ k\geq 1}} c_{i_1} t_{i_1 i_2} \cdots t_{i_{k-1} i_k}.$$

The assertion follows by some straightforward calculation, which shows that:

$$1 + \sum_{\substack{i_1<\ldots<i_k \\ k\geq 1}} e_{i_1} t_{i_1 i_2} \cdots t_{i_{k-1} i_k} = (1+\sum_l b_l)(1+\sum_{\substack{i_1<\ldots<i_k \\ k\geq 1}} c_{i_1} t_{i_1 i_2} \cdots t_{i_{k-1} i_k}).$$

Remark 5 *Let $R = \mathbf{Z}\langle\langle X\rangle\rangle$ and A_1, A_2 be matrices with coefficients in $\mathbf{Z}\langle\langle X\rangle\rangle$ where, in addition, the coefficients of A_2 (or of A_1) are polynomials. Then we always have an identity in commuting variables:*

$$\mathrm{Det}(A_1 \cdot A_2) \equiv_c \mathrm{Det}(A_1) \cdot \mathrm{Det}(A_2).$$

4 Finite sets of forbidden factors

Let $S \subseteq X^*$ denote a non-empty set being closed under factors, $S = X^* \setminus X^*FX^*$ and F is a basis of the ideal X^*FX^*. From now on we shall assume that F is finite. In particular S is a regular language being local. By l we denote the maximal length of a forbidden factor:

$$l = \max\{|f| \mid f \in F\}.$$

In order to avoid trivial special cases we shall assume $l \geq 2$ throughout. For $0 \leq p \leq q$ let $X^{[p,q]}$ denote the set

$$X^{[p,q]} = \{w \in X^* \mid p \leq |w| \leq q\}.$$

For $u \in X^*$ we denote $S(u) = \{w \in X^* \mid uw \in S\}$. Then the minimal finite deterministic automaton is given by the state space $Q' = \{S(u) \mid u \in X^*, S(u) \neq \emptyset\}$. All states are final, the initial state is $S(1)$, and for $u, v \in X^*$, $a \in X$ there exists a transition

$$S(u) \xrightarrow{a} S(v)$$

if and only if $S(ua) = S(v) \neq \emptyset$.
For a word $u \in S \cap X^{l-1}$ define

$$[u] = \{v \in X^{l-1} \mid S(u) = S(v)\}.$$

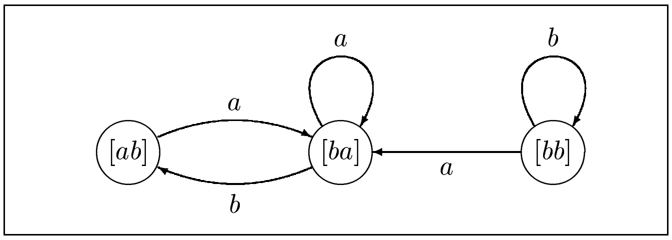

Figure 1: The automaton recognizing $\{a,b\}^* \setminus \{a,b\}^* abb\{a,b\}^*$

Let $Q = \{[u] \mid u \in S \cap X^{l-1}\}$ then we have $Q \neq \emptyset$ and we may view $Q \subseteq Q'$ by the natural mapping $[u] \longmapsto S(u)$. Note that for $[u] \in Q$ and a transition $S(u) \xrightarrow{a} S(v)$ we may take $v \in S \cap X^{l-1}$ such that $ua \in Xv$. In particular, $[v] \in Q$ and we may write $[u] \xrightarrow{a} [v]$. In the following we are concerned with the sub-automaton only which is given by the set Q.

Remark 6 *Let $n = |Q|$ and $n' = |Q'|$. Then we have $n \leq n'$, and the equality $n = n'$ is equivalent to the existence of some letter $a \in X$ such that $F \cap aX^* = \emptyset$. Indeed, if $F \cap aX^* = \emptyset$, then $S(1) = S(a^{l-1})$. Hence $Q = Q'$ in this case. For the converse let $u \in S \cap X^{l-1}$, $u = u'a$ for some $a \in X$. Let $aw \in F$. Then $w \in S$, hence $w \in S(1)$, but $w \notin S(u)$. Hence $[u] \neq [1]$.*

Example 3

i) *Let $S = \mathrm{Lex}$ as in Ex. 2, i) (I^+ being transitive). Then $n = n'$ since the first letter w. r. t. $<$ is never a prefix of a forbidden factor.*

ii) *For $F = X^l$, we have $n = 1$ and $n' = l$.*

iii) *For $F = \{abb\}$ we have $n = n' = 3$, and the automaton is given by Fig. 1.*

iv) *For $F = \{aba\}$ we have again $n = n' = 3$, and the automaton is given by almost the same picture, see Fig. 2.*

In the following we use a matrix representation for the sub-automaton generated by the set Q. Consider the following $Q \times Q$-matrix M where the coefficients are subsets of X viewed as element of $\mathbf{Z}\langle\langle X \rangle\rangle$. The coefficient $M([u],[v])$, $[u],[v] \in Q$, is written as $M[u;v]$, and defined as follows:

$$M[u;v] = \{a \in X \mid [ua] = [v]\}.$$

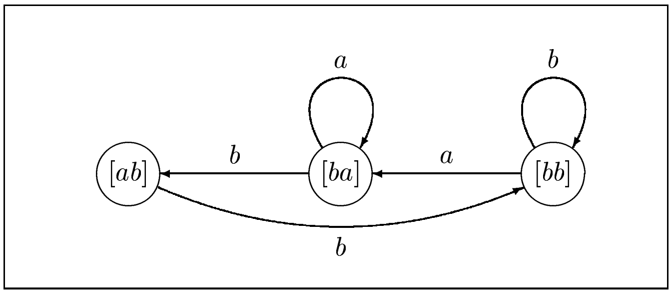

Figure 2: The automaton recognizing $\{a,b\}^* \setminus \{a,b\}^* aba \{a,b\}^*$

The symbol A is reserved to denote the following $Q \times Q$-matrix where all rows are equal:

$$A[u;v] := [v].$$

Example 4 *In both cases: $F = \{abb\}$ (Fig. 1) and $F = \{aba\}$ (Fig. 2), we have:*

$$[ab] \neq [aa] = [ba] \neq [bb] \neq [ab].$$

Therefore $Q = \{[ab],[ba],[bb]\}$ and for the representation as 3×3-matrix we may choose the ordering such that $[ab] < [ba] < [bb]$. Hence:

$$A = \begin{pmatrix} ab & aa+ba & bb \\ ab & aa+ba & bb \\ ab & aa+ba & bb \end{pmatrix}.$$

For $F = \{abb\}$ we obtain:

$$M = \begin{pmatrix} 0 & a & 0 \\ b & a & 0 \\ 0 & a & b \end{pmatrix}.$$

For $F = \{aba\}$ we have:

$$M = \begin{pmatrix} 0 & 0 & b \\ b & a & 0 \\ 0 & a & b \end{pmatrix}.$$

The next lemma is an immediate consequence of the definitions above, so we state it without proof:

Lemma 7 *Let the matrices M and A be defined as above, i.e.:*

$$M[u;v] = \{a \in X \mid [ua] = [v]\}, \quad A[u;v] = [v].$$

Let $k \geq 0$ and $[u],[v] \in Q$, then we have:

$$\begin{aligned}(AM^k)[u;v] &= \{w \in X^{(k+l-1)} \mid w \in (S \cap X^*[v])\} \\ (A\sum_{k\geq 0} M^k)[u;v] &= \{w \in X^* \mid |w| \geq l-1, w \in (S \cap X^*[v])\}.\end{aligned}$$

Proposition 8 *We have*

$$\mathrm{Det}(\mathbf{1} + A\sum_{k\geq 0} M^k) = S - (S \cap X^{[1,l-2]}).$$

Proof: By Lem. 7 all rows of the matrix $A\sum_{k\geq 0} M^k$ are equal, hence by Sect. 3, Rule 4 we obtain:

$$\begin{aligned}\mathrm{Det}(\mathbf{1} + A\sum_{k\geq 0} M^k) &= 1 + \mathrm{Trace}(A\sum_{k\geq 0} M^k) \\ &= 1 + \sum_{[v]\in Q} \{w \in X^* \mid |w| \geq l-1, w \in (S \cap X^*[v])\}.\end{aligned}$$

Moreover, for $|w| \geq l-1$ with $w = w'v'$ where $|v'| = l-1$ we have that $w \in (S \cap X^*[v])$ implies $[v] = [v']$. Hence the sum above is a partition of the set of words $\{w \in S \mid |w| \geq l-1\}$. The proposition follows.

Lemma 9 *Let $p \geq 1$. We have the following matrix identity:*

$$(\mathbf{1} + A\sum_{k\geq 0} M^k)(\mathbf{1} - M^p) = (\mathbf{1} + A(\sum_{k=0}^{p-1} M^k) - M^p).$$

Remark 10 *Since the coefficients of the matrix $(\mathbf{1} - M^p)$ are polynomials we obtain by Rem. 5 an identity in commuting variables:*

$$\mathrm{Det}(\mathbf{1} + A\sum_{k\geq 0} M^k) \cdot \mathrm{Det}(\mathbf{1} - M^p) \equiv_c \mathrm{Det}(\mathbf{1} + A(\sum_{k=0}^{p-1} M^k) - M^p).$$

Definition 4 *Let $p \geq 1$. The p-th* denominator polynomial *of S is defined by*

$$D(S,p) = \mathrm{Det}(\mathbf{1} + A(\sum_{k=0}^{p-1} M^k) - M^p) + (S \cap X^{[1,l-2]}) \cdot \mathrm{Det}(\mathbf{1} - M^p).$$

Theorem 11 *Let $p \geq 1$. We have the following identity in commuting variables:*

i)

$$S \cdot \mathrm{Det}(\mathbf{1} - M^p) \equiv_c D(S,p).$$

ii) Moreover, if

$$\mathrm{Det}((\mathbf{1} + A \sum_{k \geq 0} M^k)(\mathbf{1} - M^p)) = \mathrm{Det}(\mathbf{1} + A \sum_{k \geq 0} M^k) \cdot \mathrm{Det}(\mathbf{1} - M^p),$$

then we have the following identity of formal power series:

$$S \cdot \mathrm{Det}(\mathbf{1} - M^p) = D(S,p).$$

Proof: This is a direct consequence of Prop. 8 and Lem. 9.

The polynomial $D(S,p)$ looks complicated. However, we are mainly interested in the case $p = l - 1$. Let us rewrite

$$\begin{aligned} R &= A - M^{l-1}, \\ B &= A \cdot \sum_{k=1}^{l-2} M^k. \end{aligned}$$

Then we have for $[u], [v] \in Q$ rather simple expressions

$$\begin{aligned} R[u;v] &= \{w \in [v] \mid uw \notin S\}, \\ B[u;v] &= \{yw \in S \mid 1 \leq |y| \leq l-2, w \in [v]\}. \end{aligned}$$

Recall also that we have $A[u;v] = [v]$. Using these notations we have the interpretation:

$$D(S, l-1) = \mathrm{Det}(\mathbf{1} + R + B) + (S \cap X^{[1,l-2]})\,\mathrm{Det}(\mathbf{1} - A + R).$$

It makes sense to express the polynomial $D(S, l-1)$ in this form since A and B are matrices where all rows are equal. Note that the degree of this polynomial is at most $n \cdot (l-1) \cdot \max\{1, l-2\}$. Moreover, for $l = 2$ the matrix B is zero yielding a drastic simplification:

$$D(S,1) = \mathrm{Det}(\mathbf{1} + R).$$

The degree is then at most n.

Example 5 *i) Let $F = X^l$, then M is the zero matrix. Hence*

$$\mathrm{Det}(\mathbf{1} + A) = 1 + \mathrm{Trace}(A) = 1 + \sum_{|u|=l-1} u$$

and therefore

$$D(S,p) = 1 + (\sum_{|u|=l-1} u) + (S \cap X^{[1,l-2]}) = X^{[0,l-1]}.$$

The identity above reduces to the trivial statement

$$S = X^{[0,l-1]}.$$

ii) For $F = \{abb\}$ we have:

$$D(S,2) = 1 + a + b - abb.$$

iii) For $F = \{aba\}$ we have:

$$\begin{aligned} D(S,2) &= (a+b)(1+ab)(1-aa-bb-babb) \\ &\quad +(1+ab)[(1+ba)(1+(a+b)bb)+(aa+ba+bb)a] \\ &\equiv_c (1+ab)(1+a+b+ab+abb). \end{aligned}$$

Corollary 12 *Let $l = 2$ and $F = F_1 \cup F_2$ where $F_1 \subseteq X$ and $\emptyset \neq F_2 \subseteq XX$. Assume that there is a total order $<$ on $X \setminus F_1$ such that $ba \in F_2, a, b \in X$ implies $a < b$. Then we have $D(S,1) = 1$ and hence the Möbius function $\mu_S = S^{-1}$ is a polynomial of degree $n \leq |X \setminus F_1|$ given by following determinant:*

$$\mu_S = \mathrm{Det}(\mathbf{1} - M).$$

Proof: The matrix M is a $Q \times Q$ matrix where $Q = \{[a] \mid a \in X \setminus F_1\}$, $[a] = \{b \in X \setminus F_1 \mid \forall c \in X : ac \in F_2 \iff bc \in F_2\}$, and $n = |Q|$. Write $Q = \{[a_1], \ldots, [a_n]\}$ for some $a_1, \ldots, a_n \in X \setminus F_1$ such that $a_n < \cdots < a_1$ and $a_i \leq b$ for all $b \in [a_i]$, $1 \leq i \leq n$. We view M as an $n \times n$ matrix with $M[i;j] = M[a_i; a_j]$ for $1 \leq i, j \leq n$. We claim that the matrix R is now strictly upper triangular. Indeed, let $R[i;j] \neq \emptyset$. Then there are $b_i b_j \in F_2$ with $b_i \in [a_i]$, $b_j \in [a_j]$. Hence $a_i b_j \in F_2$ which implies $a_j \leq b_j < a_i$. Therefore $i < j$. Since R is strictly upper triangular we have $D(S,1) = D(\mathbf{1} + R) = 1$. Moreover, $\mathrm{Det}((\mathbf{1} + A + AM)(\mathbf{1} - A + R)) = \mathrm{Det}(\mathbf{1} + A + AM) \cdot \mathrm{Det}(\mathbf{1} - A + R)$ by Sect. 3, Rule 8. Hence ii) of the theorem above yields

$$S \cdot \mathrm{Det}(\mathbf{1} - M) = 1.$$

By definition of the Möbius function this shows the claim of the corollary

$$\mu_S = \mathrm{Det}(\mathbf{1} - M).$$

In particular, Thm. 11 yields:

Corollary 13 ([4], Thm. 1) *Let $<$ be a linear order of some finite alphabet X, let $I \subseteq X \times X$ be a symmetric (independence) relation, and let $\equiv_I$ be the congruence generated by $ab \equiv_I ba$ for $(a, b) \in I$. Moreover, let* Lex $\subseteq X^*$ *be the set of lexicographically first elements in each congruence class and let M be the matrix (where coefficients are subsets of X) associated to the minimal deterministic finite automaton recognizing* Lex. *Then the determinant*

$$\mathrm{Det}(\mathbf{1} - M)$$

is the formal inverse over the characteristic series over the regular set Lex *if and only if the relation $I^+ = (I\cap <)$ is transitive.*

Proof: If I^+ is transitive, then Lex is defined by the finite set $F = \{ba \mid (a, b) \in I^+\}$, hence one direction by Cor. 12. For the other direction we observe that if I^+ is not transitive, then Lex^{-1} is no polynomial, see Ex. 2 *ii)*.

5 Combinatorial interpretation

In this section we give a graph theoretical criterion when the matrix R becomes upper triangular. This yields a combinatorial interpretation in some cases.
Recall that $M^{l-1}[u; v]$ is a subset of $[v]$ for all $u, v \in S \cap X^{l-1}$ and that $[v] \subseteq S \cap X^{l-1}$. More precisely we have:

$$M^{l-1}[u; v] = \{w \in [v] \mid uw \in S\}.$$

Let us define two directed graphs associated with S. The vertex set of the graph $\widehat{G}(S)$ is defined by $\widehat{Q} = S \cap X^{l-1}$. Its edge set is given by:

$$\{(u, v) \in \widehat{Q} \times \widehat{Q} \mid uv \notin S\}.$$

The graph $G(S)$ is defined as the quotient graph of $\widehat{G}(S)$ with vertex set Q. Hence its edge set is:

$$\{([u], [v]) \in Q \times Q \mid \exists w \in [v] : uw \notin S\}.$$

A finite directed graph is acyclic if and only if there are no infinite paths. It is said to be *acyclic up to self-loops*, if every infinite path eventually becomes stationary in a loop around a single vertex.

Lemma 14

i) The graph $G(S)$ is acyclic if and only if $\widehat{G}(S)$ is acyclic.

ii) If $\widehat{G}(S)$ is acyclic up to self-loops, then $G(S)$ has this property as well.

Proof: Since $G(S)$ is a quotient graph of $\widehat{G}(S)$, it is clear that if $G(S)$ is acyclic, then $\widehat{G}(S)$ is acyclic. Hence, in order to show *i)* and *ii)* we may assume that $G(S)$ has an infinite path:

$$[v_1] \longrightarrow [v_2] \longrightarrow \cdots \longrightarrow [v_i] \longrightarrow [v_{i+1}] \longrightarrow \cdots .$$

By definition there are words $u_i', u_i \in [v_i]$ for all $1 \leq i$ such that $u_{i-1}' u_i \notin S$ for all $1 < i$. However, since $S(u_i') = S(u_i)$ we see that $u_{i-1} u_i \notin S$ for $1 < i$, too. We obtain an infinite path of $\widehat{G}(S)$:

$$u_1 \longrightarrow u_2 \longrightarrow \cdots \longrightarrow u_i \longrightarrow u_{i+1} \longrightarrow \cdots .$$

Therefore *i)*: graph $G(S)$ is acyclic if and only if $\widehat{G}(S)$ is acyclic.
Moreover, if $\widehat{G}(S)$ is acyclic up to self-loops, then for some i and all j with $i \leq j$ we have $u_i = u_j$. This implies $[v_i] = [v_j]$ for all $i \leq j$, and $G(S)$ is acyclic up to self-loops, too. This proves *ii)*.

Example 6 *Let $F \subseteq XX$ and let X be linearly ordered such that $ba \in F$ implies $a < b$. Then $\widehat{Q} = X$ and $\widehat{G}(S)$ and hence $G(S)$ are acyclic.*

Remark 15 *Let $F \subseteq X^l$ be overlap free (i. e. $uv, vw \in F$ implies $v = 1$), then $G(S)$ is acyclic.*

Proof: Assume by contradiction that $\widehat{G}(S)$ has a cycle. Then by Lem. 14 there exists a path $(u_1, u_2, \ldots, u_m)$ in $\widehat{G}(S)$ for arbitrary large m. Since F is overlap free, the word $u_1 u_2 \cdots u_m$ contains a subword of length $(m-1)l$. However the length of $u_1 u_2 \cdots u_m$ is $m(l-1)$ yielding a contradiction for $m > l$.

Example 7

i) Let $F = \{abb\}$, then $G(S)$ is acyclic. The graph $\widehat{G}(S)$ is given in Fig. 3.

ii) Let $F = \{aba\}$, then both, $\widehat{G}(S)$ and $G(S)$, are acyclic up to self-loops. They are depicted in Figs. 4 and 5

iii) Let $F = X^l$. Then $\widehat{G}(S)$ is the complete graph with $|X|^{l-1}$ vertices whereas $G(S)$ is a one point graph being trivially acyclic up to self-loops. In particular, the converse of Lem. 14, ii) is false, in general.

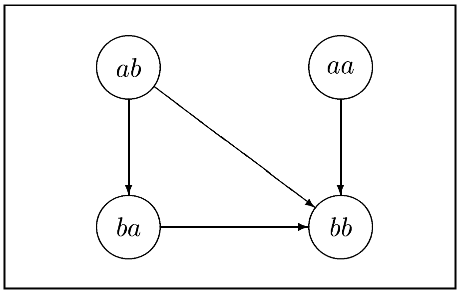

Figure 3: The graph $\widehat{G}(S)$ for $F = \{abb\}$.

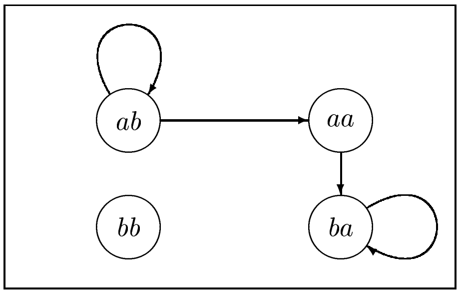

Figure 4: The graph $\widehat{G}(S)$ for $F = \{aba\}$.

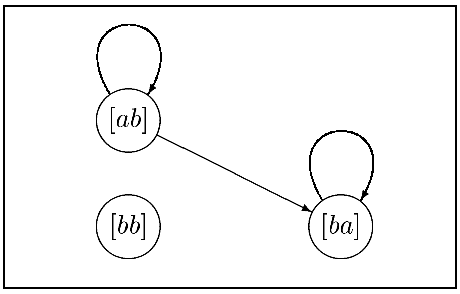

Figure 5: The graph $G(S)$ for $F = \{aba\}$.

5.1 The acyclic case: The main result

In Corollary 19 below we shall give a combinatorial interpretation for the identity of Thm. 11, ii) in the case that the graph $G(S)$ is acyclic. Recall that we have

$$M^{l-1} = A - R,$$

where $A[u;v] = [v]$ and $R[u;v] = \{w \in [v] \mid uw \notin S\}$. If $G(S)$ is acyclic, then we choose an ordering of Q such that the matrix R is strictly upper triangular.

Proposition 16 *Let $G(S)$ be acyclic. Then we have:*

$$\begin{aligned}&\mathrm{Det}(\mathbf{1} - M^{l-1})\\ &\quad = \sum_{k=0}^{n} (-1)^k \{u_1 \cdots u_k \mid u_1, \ldots, u_k \in S \cap X^{l-1}, u_{i-1}u_i \notin S \text{ for } 1 < i \leq k\}.\end{aligned}$$

Proof: Write $\mathbf{1} - M^{l-1} = \mathbf{1} - A + A - M^{l-1} = \mathbf{1} - A + R$ and apply Rule 7.

Remark 17 *The determinant is therefore the alternating sum over the (finitely many) paths of the graph $\widehat{G}(S)$.*

Example 8 *Let $F \subseteq X^3$ and let F be overlap free. Then we have*

$$\mathrm{Det}(\mathbf{1} - M^2) = 1 - X^2 + XF + FX - F^2.$$

Theorem 18 *Let $G(S)$ be acyclic, then $D(S, l-1)$ is a polynomial of degree at most $n(l-1)(l-2)$ and we have the following explicit formula*

$$\begin{aligned}D(S, l-1) \quad = \quad & 1 + \sum_{k=0}^{n} (-1)^k \{u_0 \cdots u_k \mid u_0, \ldots, u_k \in S, 1 \leq |u_0| \leq l-2,\\ & \qquad |u_i| = l-1, u_{i-1}u_i \notin S \text{ for } 1 \leq i \leq k\}.\end{aligned}$$

Proof: By definition we have

$$D(S, l-1) = \mathrm{Det}(\mathbf{1} + B + R) + (S \cap X^{[1,l-2]})\, \mathrm{Det}(\mathbf{1} - M^{l-1}).$$

Since R is strictly upper triangular we obtain by Sect. 3, Rule 7

$$\begin{aligned}
&\operatorname{Det}(\mathbf{1}+B+R)\\
&= 1+\sum_{\substack{i_1<\cdots<i_k\\ k\geq 1}}(-1)^{k-1}b_{i_1}t_{i_1,i_2}\cdots t_{i_{k-1},i_k}\\
&= 1+\sum_{k\geq 1}(-1)^{k-1}\{(u_0u_1)u_2\cdots u_k \mid 1\leq |u_0|\leq l-2, u_0u_1\in S,\\
&\qquad u_i\in S\cap X^{l-1} \text{ for } 1\leq i\leq k, u_{i-1}u_i\notin S \text{ for } 1<i\leq k\}.
\end{aligned}$$

On the other hand, Prop. 16 yields:

$$\begin{aligned}
&(S\cap X^{[1,l-2]})\operatorname{Det}(\mathbf{1}-M^{l-1})\\
&= \sum_{k\geq 0}(-1)^k\{u_0u_1\cdots u_k \mid 1\leq |u_0|\leq l-2,\\
&\qquad u_0\in S, u_i\in S\cap X^{l-1}, u_{i-1}u_i\notin S \text{ for } 1<i\leq k\}.
\end{aligned}$$

Note that both sums are finite, it is enough to sum up to $k=n$. Combining these sums we obtain:

$$\begin{aligned}
D(S,l-1) &= (S\cap X^{[0,l-2]})+\sum_{k\geq 1}(-1)^k\{u_0u_1\cdots u_k \mid 1\leq |u_0|\leq l-2,\\
&\qquad u_0,u_1,\ldots,u_k\in S, |u_i|=l-1, u_{i-1}u_i\notin S \text{ for } 1\leq i\leq k\}\\
&= 1+\sum_{k\geq 0}(-1)^k\{u_0\cdots u_k \mid u_0,\ldots,u_k\in S, 1\leq |u_0|\leq l-2,\\
&\qquad |u_i|=l-1, u_{i-1}u_i\notin S \text{ for } 1\leq i\leq k\}.
\end{aligned}$$

Example 9 *Let $F\subseteq X^3$ and let F be overlap free. Then we have:*

$$D(S,2)=1+X-F.$$

Corollary 19 *Let $G(S)$ be acyclic. Then we have the following identity of formal power series:*

$$\begin{aligned}
&S\cdot(\sum_{k\geq 0}(-1)^k\{u_1\cdots u_k \mid u_1,u_i\in S\cap X^{l-1}, u_{i-1}u_i\notin S \text{ for } 1<i\leq k\})\\
&= 1+\sum_{k\geq 0}(-1)^k\{u_0\cdots u_k \mid u_0,\ldots,u_k\in S, 1\leq |u_0|\leq l-2,\\
&\qquad |u_i|=l-1, u_{i-1}u_i\notin S \text{ for } 1\leq i\leq k\}.
\end{aligned}$$

Example 10 *Let $F \subseteq X^3$ and let F be overlap free. Then the identity above becomes*

$$S \cdot (1 - X^2 + XF + FX - F^2) = 1 + X - F.$$

Hence

$$\mu_S = 1 - X + F.$$

In particular, for $F = \{abb\}$ we reobtain

$$\mu_S = 1 - a - b + abb.$$

5.2 The case where the graph is acyclic up to self-loops

In this subsection we briefly consider the case where $G(S)$ is acyclic up to self-loops. We derive identities in commuting variables, but the formulae have a rather complicated structure. First choose an ordering of Q such that the matrix R is upper triangular. Define

$$T[u;v] = \{w \in [v] \mid uw \notin S \text{ and } [u] \neq [v]\}.$$

Let D be the diagonal matrix with

$$D[v;v] = \{w \in [v] \mid ww \notin S\}.$$

Then $R = D + T$, and T is strictly upper triangular. For index sets $J = \{j_1, \ldots, j_m\}$, $K = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ with $j_1 < \cdots < j_m$ and $i_1 < \cdots < i_k$ we denote the following finite subsets of X^* :

$$\begin{aligned} \Delta^J &= \{w_1 \cdots w_m \mid w_r w_r \notin S, w_r \in [v_{j_r}], 1 \leq r \leq m\} \\ U^K &= \{u_1 \cdots u_k \mid u_r \in [v_{i_r}], 1 \leq r \leq k, u_{r-1}u_r \notin S, 1 < r \leq k\}. \end{aligned}$$

Analogously to the formulae above we obtain the following identity in commuting variables by Thm. 11 and Rule 6:

Corollary 20 *Let $G(S)$ be acyclic up to self-loops. Then we have:*

$$\begin{aligned} &S \cdot \sum_{J \cap K = \emptyset} (-1)^{|K|} \Delta^J \cdot U^K \\ &\equiv_c \sum_{\substack{J \cap K = \emptyset \\ K = \{i_1, \ldots, i_k\} \\ i_1 < \cdots < i_k}} (-1)^k \Delta^J \cdot \{u_0 u_1 \cdots u_k \mid 0 \leq |u_0| \leq l - 2, \\ &\qquad\qquad u_0 \in S, u_r \in [v_{i_r}], u_{r-1}u_r \notin S, 1 \leq r \leq k\}. \end{aligned}$$

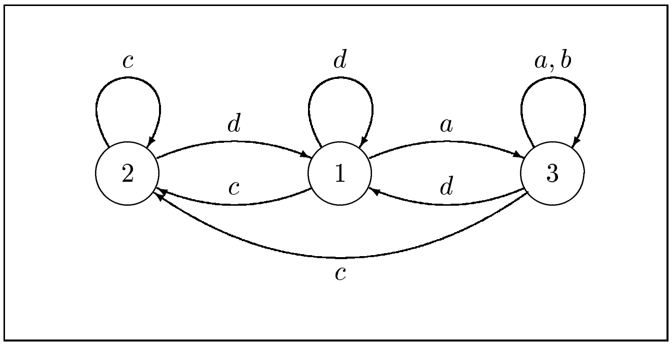

Figure 6: The automaton recognizing $\mathrm{Lex} = X^* \setminus X^*\{ca, cb, db\}X^*$ with $X = \{a, b, c, d\}$.

6 Partial commutation revisited

In this final section we consider a simple example as an illustration how our method behaves in the case of partial commutation.
Let $X = \{a, b, c, d\}$, $a < b < c < d$, and I be given by the following graph:

$$a — c — b — d.$$

Then $I^+ = (I \cap >) = \{(c, a), (c, b), (d, b)\}$ is a transitive relation and the set Lex of lexicographical normal forms is given by the set $F = \{ca, cb, db\}$ of forbidden factors, $\mathrm{Lex} = X^* \setminus X^*FX^*$.
The minimal automaton recognizing Lex has three states and is depicted in Fig. 6.
State 1 is initial, all states are final. The associated matrix $M = (M_{i,j})$ is defined such that $M_{i,j}$ denotes the sets of letters joining states i and j. Hence, we have:

$$M = \begin{pmatrix} d & c & a \\ d & c & 0 \\ d & c & a+b \end{pmatrix}.$$

Denote by A the following matrix where all rows are equal:

$$A = \begin{pmatrix} d & c & a+b \\ d & c & a+b \\ d & c & a+b \end{pmatrix}.$$

Then $A \cdot \sum_{k\geq 0} M^k$ is a matrix over formal power series where all rows are equal. Let (L_1, L_2, L_3) be any row of the matrix $A \cdot \sum_{k\geq 0} M^k$. Then it holds:

$$\begin{aligned} L_1 &= \{ud \in \mathrm{Lex} \mid u \in X^*\} \\ L_2 &= \{uc \in \mathrm{Lex} \mid u \in X^*\} \\ L_3 &= \{ux \in \mathrm{Lex} \mid u \in X^*, x \in \{a,b\}\}. \end{aligned}$$

Hence $\mathrm{Lex} = 1+L_1+L_2+L_3 = 1+\mathrm{Trace}(A\cdot\sum_{k\geq 0} M^k)$. Some direct calculation (or Rule 4) shows:

$$\mathrm{Det}(A\cdot\sum_{k\geq 0} M^k) = \mathrm{Det}\begin{pmatrix} 1+L_1 & L_2 & L_3 \\ L_1 & 1+L_2 & L_3 \\ L_1 & L_2 & 1+L_3 \end{pmatrix} = 1+L_1+L_2+L_3 = \mathrm{Lex}\,.$$

On the other hand:

$$(\mathbf{1} + A \cdot \sum_{k\geq 0} M^k)(\mathbf{1} - M) = (\mathbf{1} + A - M).$$

Now, $A - M$ is a strictly upper triangular matrix:

$$A - M = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & a+b \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $\mathrm{Det}(\mathbf{1} + A - M) = 1$. (This fact does not depend on the choice of the order of X. This determinant is 1 whether or not I^+ is transitive!).
By Rule 8 we obtain the identity on formal power series

$$\mathrm{Lex} \cdot \mathrm{Det}(\mathbf{1} - M) = 1.$$

Hence $\mathrm{Det}(\mathbf{1} - M)$ is the formal inverse of Lex. Explicitly we have:

$$\begin{aligned} \mathrm{Det}(\mathbf{1} - M) &= \mathrm{Det}\begin{pmatrix} 1-d & -c & -a \\ -d & 1-c & 0 \\ -d & -c & 1-a-b \end{pmatrix} \\ &= 1 - a - b - c - d + ca + cb + db. \end{aligned}$$

This polynomial is indeed the unambiguous lifting of the Möbius function Lex^{-1}.

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