On quadratic word equations

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Abstract. We investigate the satisfiability problem of word equations where each variable occurs at most twice (quadratic systems). We obtain various new results: The satisfiability problem is NP-hard (even for a single equation). The main result says that once we have fixed the lengths of a possible solution, then we can decide in linear time whether there is a corresponding solution. If the lengths of a minimal solution were at most exponential, then the satisfiability problem of quadratic systems would be NP-complete. (The inclusion in NP follows also from [21])
In the second part we address the problem with regular constraints: The uniform version is PSPACE-complete. Fixing the lengths of a possible solution doesn't make the problem much easier. The non-uniform version remains NP-hard (in contrast to the linear time result above). The uniform version remains PSPACE-complete.

1 Introduction

A major result in combinatorics on words states that the existential theory of equations over free monoids is decidable. This result was obtained by Makanin [14], who showed that the satisfiability of word equations with constants is decidable. For the background we refer to [19], to the corresponding chapter in the Handbook of Formal Languages, [4], or to the forthcoming [5]. There are also two volumes in the Springer lecture notes series dedicated to word equations and related topics: [23] and [2]. Makanin's Algorithm is the construction of a finite search graph. It's finiteness proof is probably among the most complex proofs in theoretical computer science. The algorithm was implemented in 1987 at Rouen by Abdulrab, see [1].
In 1990 Schulz showed an important generalization: Makanin's result remains true when adding regular constraints, [22]. Thus, we may specify for each word variable x a regular language L_x and we are only looking for solutions, where the value of each variable x is in L_x. Having this form it was also possible to extend Makanin's result to free partially commutative monoids (known also as trace monoids), see [6, 17]

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The inherent complexity of the satisfiability problem of word equations is however not well-understood. The known lower bound follows already from the unary case. A system of word equations over a unary alphabet of constants is equivalent (under logspace reductions) to an instance of linear integer programming – and vice versa. It is a well-known classical fact that linear integer programming is NP-complete, see e.g. [8]. Thus, the satisfiability problem for a single equation over a binary alphabet of constants is NP-hard. For the upper bound, a first analysis in the works of Jaffar and Schulz showed a 4-NEXPTIME result [5, 9, 24]. By Kościelski and Pacholski [10, Cor. 4.6] this went down to 3-NEXPTIME. The present state of the art is due to [7]. Gutiérrez showed that the problem is in EXPSPACE, in particular, it is in 2-DEXPTIME. Another recent and very interesting result is due to Rytter and Plandowski [21]. It shows that the minimal solution of a word equation is highly compressible in terms of Lempel-Ziv encodings. It is conjectured that the length of a minimal solution is at most exponential in the denotational length of the equation. If this were true, then the Lempel-Ziv encoding has polynomial length and, following [21], the satisfiability problem for word equation with constants would become NP-complete. At the moment we believe we are far away from proving this audacious conjecture.

The objects of interest here are quadratic systems, i.e., systems of word equations where each variable occurs at most twice. In combinatorial group theory these systems have been introduced by [12], see also [13]. They play an important rôle in the classification of closed surfaces and basic ideas of how to handle quadratic equations go back to [18]. The explicit statement of an algorithm for the solution of quadratic systems of word equations appears in [16].

We obtain various new results concerning quadratic systems. We show that the satisfiability problem is NP-hard (even for a single equation). The main result of the paper states that once we have fixed the lengths of a possible solution, then we can decide in linear time whether there is a corresponding solution. As a corollary we can say that if the lengths of a minimal solution of solvable quadratic systems were at most exponential, then the satisfiability problem would be NP-complete. The conclusion of containment in NP follows also from [21], but the method here is more direct and yields a much simpler approach to the special situation of quadratic systems.

In the second part we address the problem with regular constraints. The uniform version is PSPACE-complete. We also show that fixing the lengths of a possible solution doesn't make the problem much easier. The non-uniform version remains NP-hard (in contrast to the linear time result above). The uniform version remains PSPACE-complete.

Due to lack of space this extended abstract does not contain all proofs. They will appear elsewhere.

2 Notations and Preliminaries

Let A be an alphabet of constants and let Ω be a set of variables. As usual, $(A \cup \Omega)^*$ means the free monoid over the set $A \cup \Omega$; the empty word is denoted

by ε. A *word equation* $L = R$ is a pair $(L, R) \in (A \cup \Omega)^* \times (A \cup \Omega)^*$, and a *system* of word equations is a set of equations $\{L_1 = R_1, \ldots, L_k = R_k\}$.
A *solution* is a homomorphism $\sigma\colon (A \cup \Omega)^* \to A^*$ leaving the letters of A invariant such that $\sigma(L_i) = \sigma(R_i)$ for all $1 \le i \le k$. A solution $\sigma\colon \Omega \to A^*$ is called *minimal*, if the sum $\sum_{x \in \Omega} |\sigma(x)|$ is minimal.
A system of word equations is called *quadratic*, if each variable occurs at most twice. In the present paper we consider only quadratic systems.

3 Quadratic equations

Quadratic systems are, in principle, easy to solve by using Nielsen transformations. The standard algorithm is from [16]; it uses non-deterministic linear space and works as follows: The first step is to guess which variables can be replaced by the empty word. Then we may assume that the first equation is of the form

$$x \cdots = y \cdots$$

where $x \neq y$ and y is a variable. Moreover, we may also assume that x is a prefix of y. Then we replace all occurrences of y (at most two) by xy, and we cancel x on the left of the first equation. Having done this, we guess whether y can be replaced by the empty word. Then we repeat the process. The size of the quadratic system never increases, but the length of a minimal solution decreases in each round. Hence, the non-deterministic algorithm will find a solution, if there is any. A non-redundant execution of the algorithm will go through at most exponentially many different systems. Thus, there is an doubly exponential upper bound on the the length of the minimal solution. This seems to be quite an overestimation. We have the following conjecture.

Conjecture The length of a minimal solution of a solvable quadratic system of word equations is at most polynomial in the input size.

The value of Theorem 2 would already increase, if only the following much weaker conjecture were true.

Conjecture (weak form) The length of a minimal solution of a solvable quadratic system of word equations is at most exponential in the input size.

The first result of the present paper shows that the satisfiability problem of word equations remains NP-hard, even in the restricted case of quadratic systems. In fact, based on the conjectures above, we strongly believe that it is NP-complete in this case, see Corollary 1.

Theorem 1. *Let $|A| \ge 2$. The following problem is NP-hard.*
INSTANCE: A quadratic word equation.
QUESTION: Is there a solution $\sigma\colon \Omega \longrightarrow A^*$?

Proof. We give a reduction from 3-SAT. Let $F = C_0 \wedge \cdots \wedge C_{m-1}$ be a propositional formula in 3-CNF over a set of variables Ξ. Each clause has the form

$$C_i = (\tilde{X}_{3i} \vee \tilde{X}_{3i+1} \vee \tilde{X}_{3i+2})$$

where the $\tilde{X}_j$ are literals. We can assume that every variable has both positive and negative occurrences.
First we construct a quadratic system of words equations using word variables

$$\begin{aligned} c_i,\ d_i,\ 0 \le i \le m-1, \\ x_j,\ 0 \le j \le 3m-1, \\ y_X,\ z_X,\ \text{for each } X \in \Xi. \end{aligned}$$

We use the constants a, b, $a \neq b$. For each clause C_i we have two equations:

$$c_i x_{3i} x_{3i+1} x_{3i+2} = a^{3m} \quad \text{and} \quad c_i d_i = a^{3m-1}.$$

Now let $X \in \Xi$. Consider the set of positions $\{i_1, \ldots, i_k\}$ where $X = \tilde{X}_{i_1} = \cdots = \tilde{X}_{i_k}$ and the set of positions $\{j_1, \ldots, j_n\}$ where $\bar{X} = \tilde{X}_{j_1} = \cdots = \tilde{X}_{j_n}$. We deal with the case $k \le n$; the case $n \le k$ is symmetric. With each X we define two more equations:

$$y_X z_X = b \text{ and } x_{i_1} \cdots x_{i_k} y_X a^n b x_{j_1} \cdots x_{j_n} z_X = a^n b a^n b.$$

The formula is satisfiable if and only if the quadratic system has a solution.
Next, a system of k word equations $L_1 = R_1, \ldots, L_k = R_k$, $k \ge 1$ with $R_1 \cdots R_k \in \{a, b\}^*$ is equivalent to a single equation temporarily using a third constant c: $L_1 c \cdots L_{k-1} c L_k = R_1 c \cdots R_{k-1} c R_k$. Finally, we can eliminate the use of the third letter c without increasing the number of occurrences of any variable by the well-known technique of coding the three letters as aba, $abba$ and $abbba$ and replacing each occurrence of a variable x by axa, for a classical reference see [15].

The following theorem is the main result of the paper. In a slightly different form it appeared first in an unpublished manuscript of the first author [20].

Theorem 2. *There is a linear time algorithm to solve the following problem (on a unit cost RAM).*
INSTANCE: A quadratic system of word equations with a list of natural numbers $b_x \in \mathbb{N}$, $x \in \Omega$, written in binary.
QUESTION: Is there a solution $\sigma\colon \Omega \longrightarrow A^*$ such that $|\sigma(x)| = b_x$ for all $x \in \Omega$?

Proof. In a linear time preprocessing we can split the system into equations each containing a maximum of three variable occurrences: to see this let $x_1 \cdots x_g = x_{g+1} \cdots x_d$ be a word equation of the system with $1 \le g < d$, $x_i \in A \cup \Omega$ for $1 \le i \le d$. Then the equation is equivalent to:

$$\begin{array}{rlrl} x_1 &= y_1, & x_{g+1} &= y_{g+1}, \\ y_1 x_2 &= y_2, & y_{g+1} x_{g+2} &= y_{g+2}, \\ &\vdots & &\vdots \\ y_{g-1} x_g &= y_g, & y_{d-1} x_d &= y_d, \end{array}$$
$$y_g = y_d.$$

Here $y_1, \ldots, y_d$ denote new variables, each of them occurring exactly twice. After the obvious simplification of equations with only one variable or constant on each side, we obtain a system where each equation has the form $z = xy$, $x, y, z \in A \cup \Omega$. In fact, using (for the first time) that the lengths b_x are given, we may assume that $b_x \neq 0$ for all variables $x \in \Omega$ and that each equation has the form $z = xy$, where z is a variable. If m denotes the number of equations, then we can define the input size of a problem instance E as:

$$d(E) = m + \sum_{x \in \Omega} \log_2(b_x).$$

For $x \in A \cup \Omega$ let $|x| = b_x$ if $x \in \Omega$, and $|x| = 1$ if $x \in A$. We are looking for a solution σ such that $|\sigma(x)| = |x|$ for all $x \in A \cup \Omega$. A variable $z \in \Omega$ is called *doubly defined*, if E contains two equations $z = xy$ and $z = uv$. Let $dd(E)$ be the number of doubly defined variables. Define $c = 0.55$ and $k = 3/\ln(1/c)$ (≈ 5.01). Finally, we define the weight of the instance E as follows

$$W(E) = |\Omega| + dd(E) + k \sum_{x \in \Omega} \ln |x|.$$

We start the algorithm with the assumption that $|z| = |x| + |y|$ for all equations $z = xy$ and $|x| \neq 0$ for all variables x. Since $W(E) \in O(d(E))$, it is enough to show how to reduce the weight by at least 1 in constant time. If $dd(E) = 0$, then the system is solvable and we are done. Hence let $dd(E) > 0$. Consider a doubly defined variable z and its two equations:

$$\begin{aligned} z &= xy, \\ z &= uv. \end{aligned}$$

If $|x| = |u|$, then we either have an immediate contradiction or we can eliminate at least one variable, namely z. Therefore, without restriction, $0 < |u| < |x|$. Let w be a new variable with $|w| = |x| - |u|$. We replace the equations $z = xy$ and $z = uv$ by:

$$\begin{aligned} x &= uw, \\ v &= wy. \end{aligned}$$

If $|w| \leq c|z|$ we have reduced $\sum_{x \in \Omega} \ln |x|$ by at least $\ln(1/c)$ while increasing $dd(E)$ by at most 1. So the weight has been reduced by at least 2 and we are done with this step.
Hence in what follows we assume $|w| > c|z|$.
If $x = v$, then u and y are conjugates. Hence for some $\alpha \geq 0$ and new variables r, s we can write:

$$u = rs, w = (rs)^\alpha r, \text{ and } y = sr$$

Note that the values of $\alpha, |r|$, and $|s|$ can be calculated in constant time from $|x|$ and $|u|$. Since the system is quadratic, there are no other occurrences of x.

Hence we can replace the equations $x = uw$ and $v = wy$ by:

$$u = rs,$$
$$y = sr.$$

The overall effect is that z and x have been replaced by r and s. The number of doubly defined variables may be greater by at most 1, but we have $|r| \leq |u| < |z| - |w| \leq (1-c)|z|$ and $|s| < |x|$, so that $\sum_{x \in \Omega} \ln |x|$ has been reduced by at least $\ln(1/(1-c))$. Hence again we have reduced $W(E)$ by more than 1. Let us make a comment here: In a minimal solution we must have $\alpha = 0$. But this contradicts the assumption $|w| > c|z| > (1-c)|z| > |u|$. Hence, the case $x = v$ is not possible for a minimal solution at this stage of the algorithm.

We are in the case $x \neq v$ and $|w| > c|z|$. If neither x nor v has a second definition, then $\sum_{x \in \Omega} \ln |x|$ has not increased but the number of doubly defined variables has decreased by 1 thus decreasing $W(E)$ by 1. Hence we may assume that there is an equation

$$x = pq.$$

If p is long, which means here $|p| \geq c|z|$, then we return to the original situation:

$$z = xy,$$
$$x = pq,$$
$$z = uv.$$

We introduce a new variable r with $|r| = |z| - |p|$ and we replace the first two equations $z = xy$ and $x = pq$ by:

$$z = pr,$$
$$r = qy.$$

Since $|r| \leq (1-c)|z|$ and $|x| \geq c|z|$ we have reduced $\sum_{x \in \Omega} \ln |x|$ by at least $\ln(c/(1-c))$ while leaving $dd(E)$ unchanged (since r is not doubly defined). So $W(E)$ has decreased by more than $k \ln(c/(1-c)) > 1$ and we are done. Therefore, the situation is as follows:

$$x = uw,$$
$$x = pq,$$
$$v = wy.$$

We have $|u| < (1-c)|z|, |p| < c|z|,$ and $|w| \geq c|z|$. If $|u| = |p|$, then again, either there is an immediate contradiction or we can eliminate at least one variable. Assume first $|u| < |p|$. Then we introduce a new variable r with $|r| = |p| - |u|$ and we replace the equations $x = uw$ and $x = pq$ by:

$$p = ur,$$
$$w = rq.$$

The other (and final) case is in fact symmetric. If $|p| < |u|$, then we introduce r with $|r| = |u| - |p|$ and we replace the equations $x = uw$ and $x = pq$ by:

$$u = pr,$$
$$q = rw.$$

Overall, the number of doubly defined variables may have increased by at most 2. The variables z and x are replaced by r and w. But, in each case, we have $|r| < c|z|$ and $|w| < |x|$. Hence $\sum_{x \in \Omega} \ln |x|$ has decreased by at least $\ln(1/c)$ and so the net decrease in $W(E)$ is at least 1.
Hence, in all cases, we have decreased $W(E)$ by at least 1 in $O(1)$ arithmetic operations.

Remark 1. The method above yields a most general solution in the following sense. Let E be an instance to the problem of Theorem 2 and assume that E is solvable. Then we produce in linear time a quadratic system over a set of variables Γ (but without doubly defined variables) such that the set of solutions satisfying the length constraints is in a canonical one-to-one correspondence with the set of mappings $\psi : \Gamma \longrightarrow A^*$ where $|\psi(x)| = |b_x|$ for $x \in \Gamma$.

Corollary 1. *If the conjecture (weak form) above is true, then the satisfiability problem for quadratic systems of word equations is NP-complete.*

Remark 2. Given Theorem 1, Corollary 1 follows also from a recent work of Rytter and Plandowski [21]. They have shown that if the lengths $b_x, x \in \Omega$, are given in binary as part of the input together with a word equation (not necessarily quadratic), then there is a deterministic polynomial time algorithm for the satisfiability problem. Their method is based on Lempel-Ziv encodings and technically involved. Our contribution shows that the situation becomes much simpler for quadratic systems. In particular, we can reduce polynomial time to linear time; and our method is fairly straightforward using variable splitting. In view of the conjectures above it is not clear that the use of Lempel-Ziv encodings can improve the running time for deciding the satisfiability of quadratic systems. The most difficult part is apparently to get an idea of the lengths b_x for $x \in \Omega$. Once these lengths are known (or fixed), the corresponding satisfiability problem for quadratic systems of word equation becomes extremely simple.

4 Regular constraints

There is an interesting generalization of Makanin's result. The generalization is due to Schulz [22] and says that if a word equation is given with a list of regular languages $L_x \subseteq A^*, x \in \Omega$, then one can decide whether there is a solution $\sigma : \Omega \longrightarrow A^*$ such that $\sigma(x) \in L_x$ for all $x \in \Omega$. In the following we shall assume that regular languages are specified by non-deterministic finite automata (NFA). In the uniform version the NFA are part of the input. In the non-uniform version

the NFA are restricted such that each is allowed to have at most k states, where k is a fixed constant being not part of the input. Using a recent result of Gutiérrez [7] one can show that the uniform satisfiability problem of word equations with regular constraints can be solved in EXPSPACE (more precisely in DSPACE $(2^{o(d^3)})$, if d denotes the input size), see [5]. So, from the general case it is not really clear whether adding regular constraints makes the satisfiability problem of word equations harder. We give here however some evidence that, indeed, it does. Restricted to quadratic systems the uniform satisfiability problem with regular constraints becomes PSPACE complete.
The non-uniform version is NP-hard and it remains NP-hard, even if the lengths $b_x, x \in \Omega$, are given in unary as part of the input. This is in sharp contrast to Theorem 2. Having regular constraints it is also easy to find examples where the length of a minimal solution increases exponentially; the next example is of this kind. Note however that this refers to the uniform version of the problem

Example 1. (modified from one by [3]) Let $n > 0$. Consider the following word equation with regular constraints:

$$\begin{aligned} A = \{a, b\}, \quad \Omega &= \{x_i \mid 0 \le i \le n\}, \\ L_{x_i} &= a, \text{ for } i \le 1, \\ L_{x_i} &= aA^*a \backslash (A^* b^i A^*) \text{ for } i > 1, \\ x_n b^n x_n &= x_0 b x_0 \; b^2 x_1 b x_1 \; \cdots \; b^n x_{n-1} b^{n-1} x_{n-1} \end{aligned}$$

Theorem 3. *The following problem is PSPACE-complete.*
INSTANCE: A quadratic system of word equations with a list of regular constraints $L_x \subseteq A^*$, $x \in \Omega$.
QUESTION: Is there a solution $\sigma\colon \Omega \longrightarrow A^*$ such that $\sigma(x) \in L_x$ for all $x \in \Omega$?
Moreover, the problem remains PSPACE-complete, if the input is given together with a list of numbers b_x, $x \in \Omega$ (a number $b \in \mathbb{N}$ resp.), written in binary, and if we ask for a solution satisfying in addition the requirement $|\sigma(x)| = b_x$ ($|\sigma(x)| = b$ resp.) for all $x \in \Omega$?

Proof. The PSPACE-hardness follows directly from a well-known result on regular sets. Let $L_1, \ldots L_n$ be regular languages specified by NFA. Then the emptiness problem $L_1 \cap \cdots \cap L_n = \emptyset$ is PSPACE-complete, [11]. If the intersection is not empty, then there is a witness of at most exponential length. Let b be this upper bound on the length of a witness. Using a new letter c such that $c \notin A$, we can ask whether the intersection

$$L_1 c^* \cap \cdots \cap L_n c^*$$

contains a word of length b. (Instead of using a new letter we may also use some coding provided $|A| \ge 2$.) The quadratic system is given by n variables $x_1, \ldots, x_n$ and regular constraints $L_{x_i} = L_i c^*$ for $1 \le i \le n$. The equations are trivial: $x_1 = x_2,\ x_2 = x_3,\ \cdots,\ x_{n-1} = x_n$.
The PSPACE algorithm for the uniform satisfiability problem is a modification of the proof of Theorem 2.

Theorem 4. *Let $r \geq 4$ be a fixed constant, which is not part of the input. The following problem is NP-complete.*
INSTANCE: A quadratic system of word equations with a list of natural numbers $b_x \in \mathbb{N}$ written in binary, a list of regular constraints $L_x \subseteq A^*$, $x \in \Omega$, such that each language can be specified by some NFA of at most r states, and $|A| \geq 2$.
QUESTION: Is there a solution $\sigma\colon \Omega \longrightarrow A^*$ such that $|\sigma(x)| = b_x$ and $\sigma(x) \in L_x$ for all $x \in \Omega$?
Moreover, the problem remains NP-hard, if the numbers b_x, $x \in \Omega$, are written in unary, $|A| = 2$, and the system is a single equation.

5 Conclusion

Problems of satisfiability of quadratic word equations, with or without regular constraints, are apparently simple subcases of general problems known to be decidable. However there are a number of interesting questions still open. In the three cases studied (no constraints, uniform constraints and non-uniform constraints), we have only hardness results with no close upper bounds for the general problem where no information is given on the lengths of the solution. It would be very interesting to find a proof (or disproof!) of the conjectures of Section 3 on the minimal solution length.

References

1. Habib Abdulrab and Jean-Pierre Pécuchet. Solving word equations. *J. Symbolic Computation*, 8(5):499–521, 1990.
2. Habib Abdulrab and Jean-Pierre Pécuchet, editors. *Proceedings of Word Equations and Related Topics (IWWERT '91)*, volume 677 of *Lect. Notes Comp. Sci.* Springer-Verlag, Berlin, Heidelberg, New York, 1993.
3. Thierry Arnoux. Untersuchungen zum Makaninschen Algorithmus. Diplomarbeit 1613, Universität Stuttgart, 1998.
4. Christian Choffrut and Juhani Karhumäki. Combinatorics of words. In G. Rozenberg and A. Salomaa, editors, *Handbook of Formal Languages, Vol. 1*, pages 329–438. Springer-Verlag, Berlin, Heidelberg, New York, 1997.
5. Volker Diekert. Makanin's Algorithm. In Jean Berstel and Dominique Perrin, editors, *Algebraic Combinatorics on Words*. Cambridge University Press. A preliminary version is on the web:
http://www-igm.univ-mlv.fr/ berstel/Lothaire/index.html.
6. Volker Diekert, Yuri Matiyasevich, and Anca Muscholl. Solving trace equations using lexicographical normal forms. In P. Degano, R. Gorrieri, and A. Marchetti-Spaccamela, editors, *Proc. of the 24th ICALP, Bologna, 1997*, number 1256 in Lect. Notes Comp. Sci., pages 336–347. Springer-Verlag, Berlin, Heidelberg, New York, 1997.
7. Claudio Gutiérrez. Satisfiability of word equations with constants is in exponential space. In *Proc. of the 39th Ann. Symp. on Foundations of Computer Science, FOCS 98*, Palo Alto, California, 1998. IEEE Computer Society Press.
8. John E. Hopcroft and Jeffrey D. Ullman. *Introduction to Automata Theory, Languages, and Computation.* Addison-Wesley, Reading, Mass., 1979.

9. Joxan Jaffar. Minimal and complete word unification. *J. Assoc. Comput. Mach.*, 37(1):47–85, 1990.
10. Antoni Kościelski and Leszek Pacholski. Complexity of Makanin's algorithm. *J. Assoc. Comput. Mach.*, 43(4):670–684, 1996. Preliminary version in *Proc. of the 31st Annual IEEE Symposium on Foundations of Computer Science*, Los Alamitos, 1990.
11. Dexter Kozen. Lower bounds for natural proof systems. In *Proc. of the 18th Ann. Symp. on Foundations of Computer Science, FOCS 77*, pages 254–266. IEEE Computer Society Press, 1977.
12. Roger C. Lyndon. Equations in free groups. *Transactions of the American Mathematical Society*, 96, 1960.
13. Roger C. Lyndon and Paul E. Schupp. *Combinatorial Group Theory.* Springer-Verlag, Berlin, Heidelberg, New York, 1977.
14. Gennadií S. Makanin. The problem of solvability of equations in a free semigroup. *Mat. Sb.*, 103(2):147–236, 1977. In Russian; English translation in: *Math. USSR Sbornik, 32*, 129–198, 1977.
15. Andrei A. Markov. *Theory of Algorithms.* Academy of Sciences of the USSR, 1954. English translation, Add. Wesley 1981.
16. Yuri Matiyasevich. A connection between systems of word and length equations and Hilbert's Tenth Problem. *Sem. Mat. V. A. Steklov Math. Inst. Leningrad*, 8:132–144, 1968. In Russian; English translation in: *Seminars in Mathematics, V. A. Steklov Mathematical Institute, 8*, 61–67, 1970.
17. Yuri Matiyasevich. Some decision problems for traces. In Sergej Adian and Anil Nerode, editors, *Proceedings of the 4th International Symposium on Logical Foundations of Computer Science (LFCS'97), Yaroslavl, Russia, July 6–12, 1997*, number 1234 in Lect. Notes Comp. Sci., pages 248–257. Springer-Verlag, Berlin, Heidelberg, New York, 1997.
18. Jakob Nielsen. Die Isomorphismen der allgemeinen, unendlichen Gruppe mit zwei Erzeugenden. *Mathematische Annalen*, 78, 1918.
19. Dominique Perrin. Equations in words. In H. Ait-Kaci and M. Nivat, editors, *Resolution of equations in algebraic structures, Vol. 2*, pages 275–298. Academic Press, 1989.
20. John Michael Robson. Word equations with at most 2 occurrences of each variable. Preprint, LaBRI, Université de Bordeaux I, 1998.
21. Wojciech Rytter and Wojciech Plandowski. Application of Lempel-Ziv encodings to the solution of words equations. In Kim G. Larsen et al., editors, *Proc. of the 25th ICALP, Aarhus, 1998*, number 1443 in Lect. Notes Comp. Sci., pages 731–742. Springer-Verlag, Berlin, Heidelberg, New York, 1998.
22. Klaus U. Schulz. Makanin's algorithm for word equations: Two improvements and a generalization. In K.-U. Schulz, editor, *Proceedings of Word Equations and Related Topics, 1st International Workshop, IWWERT'90, Tübingen, Germany*, volume 572 of *Lect. Notes Comp. Sci.*, pages 85–150. Springer-Verlag, Berlin, Heidelberg, New York, 1992.
23. Klaus U. Schulz, editor. *Proceedings of Word Equations and Related Topics (IWWERT '90)*, volume 572 of *Lect. Notes Comp. Sci.* Springer-Verlag, Berlin, Heidelberg, New York, 1992.
24. Klaus U. Schulz. Word unification and transformation of generalized equations. *Journal of Automated Reasoning*, 11(2):149–184, 1993.